THE RADIUS AND MODULUS OF n-VALENCE FOR ANALYTIC FUNCTIONS WHOSE FIRST $n-1$ DERIVATIVES VANISH AT A POINT

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The principal result of this note is the determination of the precise radius and modulus of n-valence for the class of functions $f(z)=z^n+a_{n+1}z^{n+1}+\cdots$ analytic and less than or equal to M in modulus in $|z|\leqq 1$. This result readily leads to the radius and modulus of n-valence for the more general class of functions $f(z)=az^n+a_{n+1}z^{n+1}+\cdots$ analytic and less than or equal to M in modulus in $|z|\leqq R$. Finally, we note certain approximations which rather naturally suggest themselves in a search for more easily calculable constants.

We consider only expansions about the origin of functions $f(z)$ with $f(0)=0$, the generalization to expansions about a of functions $f(z)$ with $f(a)=b$ being obvious. Each circle mentioned will be understood to have the origin ($w=0$ or $z=0$) as center. The phrases *radius of n-valence* and *modulus of n-valence*, which usually refer to a class of functions, will also be used with reference to a single function. The radius of n-valence of the function $f(z)$ is the radius of the largest circle within which $f(z)$ assumes no value more than n times, and assumes at least one value n times. The modulus of n-valence of $f(z)$ is the radius of the largest circle of which the interior is covered exactly n times by the map under $f(z)$ of $|z|<\rho$, where ρ is the above radius of n-valence. Consider now one of the classes defined above. It is obvious that for each function $w=f(z)$ of the class there is a neighborhood of $z=0$ in which the function assumes no value more than n times, and assumes exactly n times every value in a sufficiently small neighborhood of $w=0$. The radius of n-valence ρ_n of the class is the radius of the largest circle within which *no* function of the class assumes a value more than n times. The modulus of n-valence m_n of the class is the radius of the largest circle of which the interior is covered exactly n times by the map of $|z|<\rho_n$ under *every* function of the class.

THEOREM. *Consider the class of functions* $f(z)=z^n+a_{n+1}z^{n+1}+\cdots$ *analytic and less than or equal to* M $(M>1)$[1] *in modulus in* $|z|\leqq 1$,

[1] The restriction to $M>1$ is necessary. By the Cauchy coefficient inequality, $M\geqq 1$, and if $M=1$ the class consists of the single function $f(z)=z^n$ for which the theorem is false.

where n and M are the constants of the class. The radius ρ_n and the modulus m_n of n-valence of the class are given by

$$m_n = \frac{M\rho_n^n(1 - M\rho_n)}{M - \rho_n}, \qquad \rho_n = M_n - (M_n^2 - 1)^{1/2},$$

where

$$M_n = \frac{1}{2}\left[\left(1 + \frac{1}{n}\right)M + \left(1 - \frac{1}{n}\right)\frac{1}{M}\right].$$

The case for $n = 1$ is completely treated in the literature. The results are due to Landau and Dieudonné.[2] The values of ρ_1 and m_1 are

$$\rho_1 = M - (M^2 - 1)^{1/2}, \qquad m_1 = M\rho_1^2 = M\rho_1 \frac{(1 - M\rho_1)}{M - \rho_1}\cdot$$

The following proof for general n consists of

(a) a proof that every function of the class is n-valent and covers $|w| < m_n$ exactly n times in $|z| < \rho_n$, and

(b) the exhibition of a single function of the class for which ρ_n and m_n are respectively the radius and modulus of n-valence.

We employ the device which Dieudonné used to find the radius of star-shapedness in the case $n = 1$ (Montel, loc. cit., p. 94), and then apply a theorem due to S. Ozaki,[3] which states that if $f(z)$ is analytic in $|z| \leqq r$ and has n zeros there, none on the circumference, and if for some real α, $\Re[e^{i\alpha}zf'(z)/f(z)] > 0$ on $|z| = r$, then $f(z)$ is n-valent in the circle $|z| < r$.

Consider

$$g(z) = f(z)/z^n = 1 + a_{n+1}z + \cdots .$$

Since $g(z)$ is analytic and less than or equal to M in modulus in $|z| \leqq 1$ with $g(0) = 1$, the following inequalities, results of the Schwarz lemma, are valid (Montel, loc. cit., p. 91):

$$(1) \qquad |g(z)| \geqq \frac{M(1 - Mr)}{M - r},$$

$$(2) \qquad |g'(z)| \leqq \frac{M^2 - |g(z)|^2}{M(1 - r^2)},$$

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[2] See Montel, *Leçons sur les Fonctions Univalentes ou Multivalentes*, 1933, pp. 90–95. This book contains a convenient collection of material relating to this paper, and we shall often refer to it.

[3] S. Ozaki, *Some remarks on the univalency and multivalency of functions*, Science Reports of Tokyo Bunrika Daigaku, section A, 2, no. 32, 1934, pp. 41–55.

where $r=|z|<1/M$. From (1) and (2),

$$\left|\frac{zg'(z)}{g(z)}\right| \leqq \frac{r(M^2-1)}{(M-r)(1-Mr)} \tag{3}$$

if $r<1/M$. But we have

$$g'(z) = \frac{z^n f'(z) - nz^{n-1}f(z)}{z^{2n}},$$

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + n. \tag{4}$$

By (3) and (4) $\Re[zf'(z)/f(z)]>0$ whenever

$$n > \frac{r(M^2-1)}{(M-r)(1-Mr)}.$$

The latter inequality reduces, for $r<1/M$, to

$$r^2 - \left[\left(1+\frac{1}{n}\right)M + \left(1-\frac{1}{n}\right)\frac{1}{M}\right]r + 1 > 0,$$

which is satisfied (factoring the left member) if

$$r < M_n - (M_n^2 - 1)^{1/2} = \rho_n,$$

where

$$M_n = \frac{1}{2}\left[\left(1+\frac{1}{n}\right)M + \left(1-\frac{1}{n}\right)\frac{1}{M}\right].$$

Now $\rho_n < 1/M$, for we shall see that ρ_n is the smallest positive zero of the derivative of

$$Mx^n(1-Mx)/(M-x),$$

and this lies between 0 and $1/M$. But from (1)

$$|f(z)| \geqq \frac{Mr^n(1-Mr)}{M-r}. \tag{5}$$

Hence $f(z)$ has precisely n zeros in $|z|<r$ and none on the boundary $|z|=r$, for every $r\leqq\rho_n$. Then:

(1) By (5), $|f(z)|\geqq m_n$ on $|z|=\rho_n$. Thus, by Rouché's theorem, $f(z)$ assumes exactly n times in $|z|<\rho_n$ every value w with $|w|<m_n$.

(2) We have seen that $\Re[zf'(z)/f(z)]>0$ on $|z|=r<\rho_n$.

Thus, by the theorem of Ozaki noted above, $f(z)$ is n-valent in the region $|z|<r$ for every $r<\rho_n$, and hence n-valent in $|z|<\rho_n$. This completes the proof of (a).

It follows from the Schwarz lemma that equality can occur in (1) for a point in $0<|z|<1$ only if $g(z)$ is of the form

$$g(z) = \frac{M(1 - Me^{i\theta}z)}{M - e^{i\theta}z} \cdot$$

We obtain the simplest function with $e^{i\theta}=1$, giving

$$f_0(z) = \frac{Mz^n(1 - Mz)}{M - z} \cdot$$

This function is of the class considered, and has ρ_n and m_n for its radius and modulus of n-valence. For,

$$f_0'(z) = \frac{M^2nz^{n-1}}{(M - z)^2}\left\{z^2 - \left[\left(1 + \frac{1}{n}\right)M + \left(1 - \frac{1}{n}\right)\frac{1}{M}\right]z + 1\right\},$$

and it is evident that the zero of $f_0'(z)$ nearest the origin (except $z=0$ itself) is precisely ρ_n. Also

$$f_0(\rho_n) = \frac{M\rho_n^n(1 - M\rho_n)}{M - \rho_n} = m_n.$$

Thus these constants are respectively the radius and modulus of n-valence for this function, and hence by (a) for the class considered in the theorem.

COROLLARY 1. *Consider the class of functions $f(z)=az^n+a_{n+1}z^{n+1}+\cdots$ analytic and less than or equal to M $(M>|a|R^n)$ in modulus in $|z|\leq R$, the constants of the class being $|a|$ $(\neq 0)$, n, M and R. The radius ρ_n and the modulus m_n of n-valence of the class are given by*

$$\rho_n = R\sigma, \qquad m_n = \frac{M\sigma^n(|a|R^n - M\sigma)}{M - |a|R^n\sigma},$$

where σ is defined by

$$\sigma = M_n - (M_n^2 - 1)^{1/2},$$

$$M_n = \frac{1}{2}\left[\left(1 + \frac{1}{n}\right)\frac{M}{|a|R^n} + \left(1 - \frac{1}{n}\right)\frac{|a|R^n}{M}\right].$$

This corollary follows immediately from the theorem upon considering the function $g(z)=f(Rz)/aR^n$.

The results on the modulus of n-valence relate to a result due to Walsh and Seidel,[4] who do not assume $f'(0) = \cdots = f^{(n-1)}(0) = 0$, but who, on the other hand, do not obtain the sharp inequality.

COROLLARY 2. *For the class of functions of Corollary* 1

$$\rho_n > \frac{|a| R^{n+1}}{2M}, \qquad m_n > M\left(\frac{|a| R^n}{2M}\right)^{n+1}.$$

For, $M/|a|R^n > 1$, from which it follows that $M/|a|R^n \geqq M_n > 1$. Therefore, since

$$M_n - (M_n^2 - 1)^{1/2} = \frac{1}{M_n + (M_n^2 - 1)^{1/2}} > \frac{1}{2M_n},$$

we find that

$$\frac{|a| R^{n+1}}{2M} < \rho_n. \tag{6}$$

Moreover, if $f(z) = Mz^n(|a|R^{n+1} - Mz)/(MR^{n+1} - |a|R^{2n}z)$, then $f(\rho_n) = m_n$, $f'(\rho_n) = 0$, $f(|a|R^{n+1}/2M) < m_n$ by (6), and the last inequality easily gives

$$M\left(\frac{|a| R^n}{2M}\right)^{n+1} < m_n. \tag{7}$$

But (6) and (7) constitute the corollary.

The inequality (7) is an improvement over an approximation due to Privaloff,[5] which states that the image of the circle $|z| < R$ under the function $w = f(z)$ of the class of Corollary 1 covers at least n times the circle

$$|w| < \frac{8}{3} M\left(\frac{R^n |a|}{4M}\right)^{n+1}.$$

For,

$$M\left(\frac{|a| R^n}{2M}\right)^{n+1} > \frac{8}{3} M\left(\frac{|a| R^n}{4M}\right)^{n+1},$$

and by (7), the circle $|w| < M(|a|R^n/2M)^{n+1}$ is covered exactly n times by the image of $|z| < \rho_n$ and hence at least n times by that of $|z| < R$.

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[4] Walsh and Seidel, *On the derivatives of functions analytic in the unit circle*, Proceedings of the National Academy of Sciences, vol. 24 (1938), pp. 337–340.

[5] J. Privaloff, *Sur un théorème de M. Bloch*, Recueil Mathématique de Moscou, vol. 35 (1928), pp. 111–121.

We conclude with a few miscellaneous remarks.

1. The modulus of n-valence for the class of the theorem is also the radius of the largest circle within which converge all the power series (in $w^{1/n}$) of the inverse functions.

2. It is evident from the proof of part (b) of the theorem that the radius and modulus of n-valence for the function $f(z)$ are ρ_n and m_n only if

$$f(z) = \frac{Mz^n(1 - Me^{i\theta}z)}{M - e^{i\theta}z}$$

for some real θ. For any other function of the class the radius and modulus of n-valence are greater, respectively, than ρ_n and m_n.

3. The inequalities $\leqq M$, $\leqq 1$ of the theorem may be replaced by $< M$, < 1 without affecting the validity of the work.

4. It is easily seen that the equations of Corollary 1 give, for instance, the radius of n-valence for the class of functions of the form $f(z) = a_f z^n + a_{n+1}z^{n+1} + \cdots$ analytic and less than or equal to M_f in modulus in $|z| \leqq R$, including just those functions for which $M_f/|a_f|$ is less than some preassigned bound ($M/|a|$ of the equations of the corollary). No modulus of n-valence exists for this class.

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