ON THE SUPPORTING-PLANE PROPERTY OF A CONVEX BODY[1]

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In an earlier paper,[2] the authors have shown that in a linear space $\mathfrak{S}$ with an inner product, a set $\mathfrak{M}$ which is closed and linearly connected is supported at a set of boundary points which is everywhere dense on the boundary of $\mathfrak{M}$, and an example is given to show that such a set $\mathfrak{M}$ may have boundary points through which no supporting plane exists. The purpose of this paper is to show that if a set, in addition to being linearly connected and closed, also possesses inner points, then it is completely supported at its boundary points. In (I), reference was made to a paper by Ascoli in which such a result was obtained in a separable space. We do not assume our space $\mathfrak{S}$ to be separable. The definitions and results of (I) will be used in this paper.

A set $\mathfrak{K}$, which is a proper subset of the space $\mathfrak{S}$, will be called a *convex body* if it is linearly connected, closed, and possesses inner points. In the sequel $\mathfrak{K}$ will always denote a convex body.

With reference to the set $\mathfrak{K}$, there is associated with each point x of the space $\mathfrak{S}$ a nonnegative number $r(x)$: if x is an inner point of $\mathfrak{K}$, $r(x)$ is defined as the least upper bound of the radii of spheres about x which do not contain points exterior to $\mathfrak{K}$; for other points of $\mathfrak{S}$, $r(x)$ is defined to be zero. We will call $r(x)$ *the radius at the point* x.

If x_1 is a point of $\mathfrak{K}$, all points x of the sphere $\|x-x_1\| \leqq r(x_1)$ are points of $\mathfrak{K}$.

THEOREM 1. *Let r_1 and r_2 be the radii at the points x_1 and x_2, respectively, of the convex body $\mathfrak{K}$. Then the radius r at the point*

$$x = x_1 + k(x_2 - x_1), \qquad 0 \leqq k \leqq 1,$$

satisfies

$$r \geqq r_1 + k(r_2 - r_1).$$

PROOF. Let $y=x+\rho u$, where $\rho=r_1+k(r_2-r_1)$ and $\|u\|=1$. The points $y_1=x_1+r_1u$ and $y_2=x_2+r_2u$ are points of $\mathfrak{K}$. But from the definitions of x, ρ, and y, it follows that $y=y_1+k(y_2-y_1)$. Hence y, being on the segment joining y_1 and y_2, is also a point of $\mathfrak{K}$. Consequently

[1] Presented to the Society, September 5, 1939.

[2] *On convexity in a linear space with an inner product*, Duke Mathematical Journal, vol. 5 (1939), pp. 520–534. Hereafter, this paper will be referred to by (I).

all points on the boundary of the sphere with radius ρ and center x are in $\mathfrak{K}$. Since $\mathfrak{K}$ is linearly connected, also all points within this sphere are in $\mathfrak{K}$. Therefore $r \geqq \rho$ and the theorem is established.

The following corollaries, which appear self-evident in ordinary space, can be shown to be direct consequences of the preceding theorem.

COROLLARY 1. *Each point of the segment joining two inner points of $\mathfrak{K}$ is an inner point of $\mathfrak{K}$.*

COROLLARY 2. *If x_0 is a boundary point and x_1 an inner point of $\mathfrak{K}$, then the points $x = x_1 + k(x_0 - x_1)$ are inner points of $\mathfrak{K}$ for $0 \leqq k < 1$, and exterior points for $k > 1$.*

With reference to a given boundary point x_0 of the set $\mathfrak{K}$, there is associated with each point x, other than x_0, of the space $\mathfrak{S}$ a non-negative number $s(x)$, defined by[3]

$$s(x) = r(x)/\|x - x_0\|.$$

If x is an exterior point or a boundary point of $\mathfrak{K}$, other than x_0, $s(x)$ is equal to zero; if x is an inner point of $\mathfrak{K}$, $s(x)$ is positive; $s(x_0)$ is not defined.

It is also obvious that $s(x) \leqq 1$, since $r(x) \leqq \|x - x_0\|$.

THEOREM 2. *Let x_0 be a given boundary point of the convex body $\mathfrak{K}$, and let x_t be given by*

$$x_t = x_0 + tu, \qquad \text{where } t > 0,\ \|u\| = 1. \tag{1}$$

Then, for fixed u,

(a) *$s(x_t)$ is a non-decreasing function as $t \to 0$; and*

(b) *$\lim_{t\to 0} s(x_t)$ exists.*

PROOF. In case there are no points of $\mathfrak{K}$ given by (1), the theorem is obviously true, for then

$$s(x_t) = 0 \quad \text{for} \quad t > 0, \qquad \lim_{t\to 0} s(x_t) = 0.$$

In case there are points of $\mathfrak{K}$ given by (1), let x_1 and x_2 be two points of $\mathfrak{K}$ on (1) for parameter values t_1 and t_2, where $t_1 < t_2$; then we have

$$\begin{aligned} x_1 &= x_0 + t_1 u, & x_2 &= x_0 + t_2 u; \\ s(x_1) &= r(x_1)/t_1, & s(x_2) &= r(x_2)/t_2. \end{aligned} \tag{2}$$

[3] Since s is a function of x_0 as well as x, a more explicit notation would be $s(x_0, x)$; but the simpler notation will suffice, inasmuch as the function is to be used in the sequel only with reference to a fixed boundary point x_0.

But $x_1=x_0+(t_1/t_2)(x_2-x_0)$, and hence, by Theorem 1, we have

$$r(x_1) \geqq \frac{t_1}{t_2} r(x_2),$$

since $r(x_0)=0$. Therefore, by (2), $s(x_1)\geqq s(x_2)$.

This result establishes part (a) of the theorem. Since $s(x_t)$ cannot exceed one, obviously part (b) of the theorem is true.

Let Σ be the unit sphere about x_0; and let p_u be the point on Σ given by $p_u=x_0+u$, $\|u\|=1$. Let $x_t=x_0+tu$, $(0<t<1)$, be the segment joining x_0 to p_u; and let[4]

$$\sigma(u) = \lim_{t\to 0} s(x_t).$$

We thus have a function $\sigma(u)$ uniquely defined at each point p_u on the sphere Σ. Obviously, by its definition, we have

$$0 \leqq \sigma(u) \leqq 1.$$

Also $\sigma(u)=0$ only if the segment joining x_0 to p_u does not contain any inner points of $\mathfrak{K}$. If the segment joining x_0 to p_u contains inner points of $\mathfrak{K}$, we have $\sigma(u)>0$.

LEMMA 1. *Let p_u and p_v be two points on Σ, such that*

$$p_u = x_0 + u, \qquad p_v = x_0 + v, \qquad v = - u.$$

Then at least one of the numbers $\sigma(u)$ or $\sigma(v)$ is equal to zero.

PROOF. Assume $\sigma(u)>0$; then the segment joining x_0 to p_u contains inner points. Consequently, by Corollary 2, the segment joining x_0 to p_v does not contain any inner points. Therefore, $\sigma(v)=0$.

THEOREM 3. *Let x_0 be a given boundary point of the convex body $\mathfrak{K}$, and let Σ be the unit sphere about x_0. Let p_u and p_v given by*

$$p_u = x_0 + u, \qquad \|u\| = 1, \qquad p_v= x_0 + v, \qquad \|v\| = 1$$

be two distinct points on Σ, for which $\sigma(u)$ and $\sigma(v)$ are both positive. Then there exists a point p_w distinct from p_u and p_v for which

$$\sigma(w) > \tfrac{1}{2}[\sigma(u) + \sigma(v)].$$

[4] The limit was shown to exist in Theorem 2; we are denoting the value of this limit by $\sigma(u)$. It may be of interest to note that

$$\sigma(u) = \lim_{x\to x_0,\ \text{along } x_t=x_0+tu} s(x) = \lim_{x\to x_0} \frac{r(x) - r(x_0)}{\|x - x_0\|} = r'_u(x_0)$$

is the *directional derivative* of $r(x)$ at x_0 in the direction u.

PROOF. Let x_t and y_t be points of $\mathfrak{K}$ given by

$$x_t = x_0 + tu, \qquad y_t = x_0 + tv, \qquad 0 < t < 1,$$

and let $((u, v)) = \lambda$. Then, certainly $|\lambda| \leqq 1$. But if $\lambda = 1$, $u = v$, and p_u and p_v are not distinct. If $\lambda = -1$, $u = -v$, in which case not both of the numbers $\sigma(u)$ and $\sigma(v)$ can be positive, because of Lemma 1. Consequently, we have

$$-1 < \lambda < 1.$$

Let $z_t = \frac{1}{2}(x_t + y_t)$; then $z_t = x_0 + \xi t w$, where $\|w\| = 1$ and $\xi = \frac{1}{2}(1+\lambda)^{1/2}$. Thus

$$0 < \xi < 1.$$

We thus have a point p_w on the sphere Σ defined by $p_w = x_0 + w$. Now, $r(z_t) \geqq \frac{1}{2}[r(x_t) + r(y_t)]$, by Theorem 1. Hence

$$s(z_t) = \frac{r(z_t)}{\xi t} \geqq \frac{1}{2\xi}\left[\frac{r(x_t)}{t} + \frac{r(y_t)}{t}\right] = \frac{1}{2\xi}\left[s(x_t) + s(y_t)\right],$$

and

$$\lim_{t\to 0} s(z_t) \geqq \frac{1}{2\xi} \lim_{t\to 0} \left[s(x_t) + s(y_t)\right],$$

from which

$$\sigma(w) \geqq \frac{1}{2\xi}\left[\sigma(u) + \sigma(v)\right] > \frac{1}{2}\left[\sigma(u) + \sigma(v)\right].$$

Thus the theorem is established.

Let $\bar{\sigma}$ denote the least upper bound of the function $\sigma(u)$ as p_u varies over the sphere Σ. Then, also $0 \leqq \bar{\sigma} \leqq 1$; and $\bar{\sigma} = 0$ is possible only for sets which do not have any inner points. For a convex body $\mathfrak{K}$, we have $0 < \bar{\sigma} \leqq 1$.

In the material which follows, it is to be understood that x_0 is a *fixed* boundary point of the convex body $\mathfrak{K}$, $s(x)$ is defined relative to x_0, Σ is the unit sphere about x_0, $\sigma(u)$ is the function defined above on the boundary of Σ, and $\bar{\sigma}$ the least upper bound of $\sigma(u)$ on Σ.

THEOREM 4. *If there is a point p_u on Σ for which $\sigma(u) = \bar{\sigma}$, this point is unique.*

PROOF. Suppose, if possible, that there were a second point p_v for which $\sigma(v) = \bar{\sigma}$. Then, by Theorem 3, since $\bar{\sigma} > 0$, there would be a point p_w for which

$$\sigma(w) > \tfrac{1}{2}[\sigma(u) + \sigma(v)] = \bar{\sigma}.$$

But since no $\sigma(w)$ can exceed $\bar{\sigma}$, there cannot be a second point p_v of the type described.

THEOREM 5. *Let p_u be a point on Σ for which $\sigma(u)=\bar{\sigma}$. If v satisfies the conditions $\|v\|=1$ and $((u, v))<0$, then the points $z_t=x_0+tv$, $t>0$, are exterior points of $\mathfrak{K}$.*

PROOF. Let $p_u=x_0+u$, $\|u\|=1$, and $p_v=x_0+v$, $\|v\|=1$; and let $((u, v))=-\lambda$, where $\lambda>0$. Assume, if possible, that there is a point $z=x_0+dv$, $d>0$, belonging to $\mathfrak{K}$. Let w be the projection (defined in (I)) of z on the line through x_0 and p_u. Then

$$w = p_u + c(x_0 - p_u),$$

where

$$\begin{aligned} c &= \frac{((z - p_u, x_0 - p_u))}{\|p_u - x_0\|^2} = ((z - x_0 - u, - u)) \\ &= ((dv - u, - u)) = 1 + \lambda d. \end{aligned}$$

Hence,

$$w = p_u - (1 + \lambda d)u = x_0 - \lambda du.$$

On the segment joining x_0 to p_u, let $x_t=x_0+tu$ be an inner point of $\mathfrak{K}$. Let y_t be the projection of x_0 on the line through x_t and z. Then

$$y_t = x_t + k(z - x_t)$$

where

$$k = \frac{((x_0 - x_t, z - x_t))}{\|z - x_t\|^2} = \frac{((- tu, dv - tu))}{\|z - x_t\|^2} = \frac{\lambda td + t^2}{d^2 + 2\lambda td + t^2},$$

since $z-x_t=z-x_0+x_0-x_t=dv-tu$ and

$$\|z - x_t\|^2 = d^2 - 2td((u, v)) + t^2 = d^2 + 2\lambda td + t^2.$$

From the above value of k, it is easily seen that $0<k<1$, which means that y_t is a point of $\mathfrak{K}$. The following are easily established:

$$\|y_t - x_0\|^2 = \frac{t^2d^2(1 - \lambda^2)}{d^2 + 2\lambda td + t^2} \neq 0,$$

since $\lambda \neq \pm 1$, and

$$\|z - w\|^2 = d^2(1 - \lambda^2).$$

From these, and previous results, we obtain

$$\frac{\|x_t - x_0\|^2}{\|y_t - x_0\|^2} = \frac{t^2(d^2 + 2\lambda td + t^2)}{t^2d^2(1 - \lambda^2)} = \frac{d^2 + 2\lambda td + t^2}{d^2(1 - \lambda^2)} = \frac{\|z - x_t\|^2}{\|z - w\|^2}.$$

Therefore, we have

$$\frac{\|x_t - x_0\|}{\|y_t - x_0\|} = \frac{\|z - x_t\|}{\|z - w\|}. \tag{3}$$

Now $s(y_t) = r(y_t)/\|y_t - x_0\|$ and $s(x_t) = r(x_t)/\|x_t - x_0\|$, where $r(y_t)$ and $r(x_t)$ denote the radii at the points y_t and x_t, respectively. Also $r(y_t) \geqq (1-k)r(x_t)$, by Theorem 1 and the definition of y_t. Hence

$$\begin{aligned} \frac{s(y_t)}{s(x_t)} = \frac{r(y_t)}{\|y_t - x_0\|} \cdot \frac{\|x_t - x_0\|}{r(x_t)} &\geqq (1 - k)\frac{\|x_t - x_0\|}{\|y_t - x_0\|} \\ &= (1 - k)\frac{\|z - x_t\|}{\|z - w\|}, \end{aligned} \tag{4}$$

the last equality being a consequence of (3).

But $k = \|y_t - x_t\|/\|z - x_t\|$ and $1 - k = \|z - y_t\|/\|z - x_t\|$. Therefore, from (4), we have

$$s(y_t) \geqq \frac{\|z - y_t\|}{\|z - w\|} s(x_t). \tag{5}$$

Now,

$$\lim_{t \to 0} \frac{\|z - y_t\|}{\|z - w\|} = \frac{d}{d(1 - \lambda^2)^{1/2}} = \frac{1}{(1 - \lambda^2)^{1/2}} > 1,$$

since z and w are independent of t, while $y_t \to x_0$ as $t \to 0$. Therefore, from (5),

$$\lim_{t \to 0} s(y_t) \geqq \frac{1}{(1 - \lambda^2)^{1/2}} \sigma(u) > \sigma(u) = \bar{\sigma}.$$

But this is impossible; hence the assumption that z was a point of $\mathfrak{K}$ is untenable.

THEOREM 6. *Let p_u be a point on Σ for which $\sigma(u) = \bar{\sigma}$. Then the plane*

$$\pi(x) \equiv ((u, x - x_0)) = 0 \tag{6}$$

is a supporting plane of $\mathfrak{K}$ through the boundary point x_0.

PROOF. If the plane (6) were not a supporting plane, there would be a point z of $\mathfrak{K}$ for which $\pi(z) < 0$. Let $v = (z - x_0)/\|z - x_0\|$; then

$$((u, v)) = \frac{\pi(z)}{\|z - x_0\|} < 0, \qquad z = x_0 + \|z - x_0\| v.$$

But, v satisfies the conditions of Theorem 5; therefore z must be an exterior point of $\mathfrak{K}$. Consequently, there cannot be a point z of $\mathfrak{K}$ for which $\pi(z) < 0$; and (6) is indeed a supporting plane.

THEOREM 7. *Let x_0 be a given boundary point of the convex body $\mathfrak{K}$, and let Σ be the unit sphere about x_0. There is a unique point $p_{\bar{u}}$ on Σ for which $\sigma(\bar{u}) = \bar{\sigma}$.*

PROOF. We have only to show the existence of one point $p_{\bar{u}}$ for which $\sigma(\bar{u}) = \bar{\sigma}$. The uniqueness of this point will follow from Theorem 4.

From the definition of $\bar{\sigma}$ it follows that for any preassigned $\epsilon > 0$, there exists a point on Σ for which the value of σ is greater than $\bar{\sigma} - \epsilon$. Choose a monotone decreasing sequence of positive numbers $\{\epsilon_n\}$ with limit zero. Corresponding to each ϵ_n there exists a point p_{u_n} on Σ for which $\sigma(u_n) > \bar{\sigma} - \epsilon_n$. We wish to show that the sequence of points $\{p_{u_n}\}$ on Σ converges.

Let $p_{u_n} = x_0 + u_n$, $\|u_n\| = 1$, and $p_{u_m} = x_0 + u_m$, $\|u_m\| = 1$. Then

$$\|p_{u_n} - p_{u_m}\|^2 = 2 - 2((u_n, u_m)). \tag{7}$$

Let $w = \frac{1}{2}(1/\xi)(u_n + u_m)$, where ξ is so chosen that $\|w\| = 1$. Then we have

$$\xi^2 = \tfrac{1}{2}[1 + ((u_n, u_m))]. \tag{8}$$

Let $p_w = x_0 + w$; from the proof of Theorem 3, we know that

$$\sigma(w) \geqq \frac{1}{2\xi}[\sigma(u_n) + \sigma(u_m)] > \frac{1}{2\xi}[2\bar{\sigma} - \epsilon_n - \epsilon_m].$$

But $\bar{\sigma} \geqq \sigma(w)$; hence $\bar{\sigma} > (1/\xi)[\bar{\sigma} - (\epsilon_n + \epsilon_m)/2]$, from which

$$\xi^2 > \left[1 - \frac{1}{2\bar{\sigma}}(\epsilon_n + \epsilon_m)\right]^2.$$

Using the value of ξ^2 from (8) we easily find that

$$((u_n, u_m)) > 2\left[1 - \frac{1}{2\bar{\sigma}}(\epsilon_n + \epsilon_m)\right]^2 - 1.$$

Then using (7), we obtain

$$(9) \qquad \|p_{u_n} - p_{u_m}\|^2 < \frac{4}{\bar\sigma}(\epsilon_n + \epsilon_m) - \frac{1}{\bar\sigma^2}(\epsilon_n + \epsilon_m)^2.$$

Since $\lim_{n,m\to\infty} \|p_{u_n} - p_{u_m}\| = 0$ and the space $\mathfrak{S}$ is complete, as was assumed in (I) and throughout this paper, the sequence $\{p_{u_n}\}$ converges to a point $p_{\bar u}$. This point $p_{\bar u}$ is on Σ, and moreover $\sigma(\bar u) = \bar\sigma$, since it is easily shown that $\sigma(\bar u)$ is greater than $\bar\sigma - \epsilon$ for any preassigned positive ϵ.

THEOREM 8. *A convex body $\mathfrak{K}$ is completely supported at its boundary points.*

PROOF. Let x_0 be a boundary point of $\mathfrak{K}$. There exists a point p_u on the unit sphere Σ about x_0 for which $\sigma(u) = \bar\sigma$, by Theorem 7. Hence the plane $((u, x - x_0)) = 0$ is a supporting plane of $\mathfrak{K}$ through x_0, by Theorem 6. Since similar statements can be made for each boundary point, $\mathfrak{K}$ is completely supported at its boundary points.

From the material above, the following additional result may be established without much difficulty:

COROLLARY 3. *There exists a unique supporting plane through the boundary point x_0 of the convex body $\mathfrak{K}$ only if $\bar\sigma = 1$; for $\bar\sigma < 1$, there is an infinite number of supporting planes through x_0.*

A primary classification of boundary points of a convex body may thus be made in terms of $\bar\sigma$, which is a function defined over the boundary of the convex body.

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