CERTAIN SELF-RECIPROCAL FUNCTIONS

BRIJ MOHAN

In 1932 and 1933 I [5, 6] gave some rules connecting different classes of self-reciprocal functions. The object of this note is to derive some new self-reciprocal functions with the help of those rules.

I will say that a function is R_ν if it is self-reciprocal for J_ν transforms, where $\nu > -1$.

I will make use of the following results given in the papers referred to:

If $f(x)$ is R_μ, the functions $g(x)$ given by the following integral formulas are all R_ν:

(i) $$g(x) = x^{(\nu-\mu+1)/2}\int_0^\infty y^{(\nu-\mu+1)/2}J_{(\mu+\nu)/2}(xy)f(y)dy,$$

(ii) $$g(x) = x^{(\mu-\nu+1)/2}\int_0^\infty y^{(\mu-\nu+1)/2}J_{(\mu+\nu)/2}(xy)f(y)dy,$$

(iii) $$g(x) = \int_0^\infty \frac{y^{\mu+1/2}f(xy)}{(1+y^2)^{1+\mu/2+\nu/2}}dy,$$

(iv) $$g(x) = \int_1^\infty \frac{y^{1/2-\mu}f(xy)}{(y^2-1)^{1-\mu/2+\nu/2}}dy,$$

(v) $$g(x) = \int_0^1 \frac{y^{1/2+\mu}f(xy)}{(1-y^2)^{1+\mu/2-\nu/2}}dy.$$

If, in (ii) we take the familiar R_μ function

(1) $$x^{\mu+1/2}e^{-x^2/2}$$

for $f(x)$, we get

$$g(x) = x^{(\mu-\nu+1)/2}\int_0^\infty y^{(\mu-\nu+1)/2}J_{(\mu+\nu)/2}(xy)\cdot y^{\mu+1/2}e^{-y^2/2}dy$$

$$= x^{(\mu-\nu+1)/2}\int_0^\infty y^{3\mu/2-\nu/2+1}e^{-y^2/2}J_{\mu/2+\nu/2}(xy)dy.$$

Evaluating this integral by Hankel's formula [7], we get

$$g(x) = x^{(\mu-\nu+1)/2}\frac{\Gamma(\mu+1)(x/2^{1/2})^{\mu/2+\nu/2}}{2^{-3\mu/4+\nu/4}\Gamma(1+\mu/2+\nu/2)}e^{-x^2/2}$$
$$\cdot {}_1F_1(\nu/2-\mu/2;\,1+\mu/2+\nu/2;\,x^2/2), \qquad \mu > -1.$$

This shows that the function $x^{\mu+1/2}e^{-x^2/2}\,{}_1F_1(\nu/2-\mu/2;1+\mu/2+\nu/2;x^2/2)$, $(\mu>-1)$, which is the same as

(2) $$x^{\nu+2n+1/2}e^{-x^2/2}\,{}_1F_1(-n;n+\nu+1;x^2/2),\qquad \nu+2n>-1,$$

is R_ν.

Particular cases. (α) Since, when n is a positive integer,

$$L_n^{(\alpha)}(x)=\frac{\Gamma(1+\alpha+n)}{n!\Gamma(\alpha+1)}\,{}_1F_1(-n;\alpha+1;x),$$

where $L_n^{(\alpha)}(x)$ denotes the generalized Laguerre polynomial of order n, it follows that the function $x^{\nu+2n+1/2}e^{-x^2/2}L_n^{(n+\nu)}(x^2/2)$, $(\nu+2n>-1)$, is R_ν for positive integral values of n.

This result was given by Howell [**4**].

(β) Since, when n is a positive integer,

$$T_\mu^n(x)=\frac{(-1)^n}{n!\Gamma(1+\mu)}\,{}_1F_1(-n;1+\mu;x),$$

where $T_\mu^n(x)$ is Sonine's polynomial of order n, it follows that the function

$$x^{\nu+2n+1/2}e^{-x^2/2}T_{n+\nu}^n(x^2/2),\qquad \nu+2n>-1,$$

is R_ν for positive integral values of n.

(γ) If we put $n=-\nu-1/2$, our function becomes $x^{-\nu-1/2}e^{-x^2/2}\,{}_1F_1(\nu+1/2;1/2;x^2/2)$.

As a particular case, when ν is an integer, this becomes a constant multiple of

(3) $$x^{-\nu-1/2}e^{-x^2/4}D_{-2\nu-1}(x).$$

(δ) When $n=1/2-\nu$, the function is $x^{-\nu+3/2}e^{-x^2/2}\,{}_1F_1(\nu-1/2;3/2;x^2/2)$, $(\nu<2)$.

If ν is an integer, this is a constant multiple of

(4) $$x^{-\nu+1/2}e^{-x^2/4}D_{-2\nu+1}(x).$$

The functions (3) and (4) were given by Mitra [**2**].

Taking the same function (1) for $f(x)$, we have, from (iii),

$$g(x)=\int_0^\infty \frac{y^{\mu+1/2}e^{-x^2y^2/2}(xy)^{\mu+1/2}}{(1+y^2)^{1+\mu/2+\nu/2}}\,dy=x^{\mu+1/2}\int_0^\infty\frac{y^{2\mu+1}e^{-x^2y^2/2}}{(1+y^2)^{1+\mu/2+\nu/2}}\,dy.$$

Evaluating this integral by a formula given by Whittaker [**8**] we find that $g(x)$ is a constant multiple of

$$x^{(\mu+\nu-1)/2}e^{x^2/4}W_{-1/2-3\mu/4-\nu/4,(\nu-\mu)/4}(x^2/2),$$

where $\mu > -1$.

Thus, we see that the function

$$x^{\nu-2m-1/2}e^{x^2/4}W_{3m-\nu-1/2,m}(x^2/2), \tag{5}$$

where $\nu - 4m > -1$, is R_ν.

This function was given by Bailey [1].[1]

If we put the same function (1) for $f(x)$ in (i) and evaluate the corresponding integral by Hankel's formula [7], we arrive back at the R_ν function

$$x^{\nu+1/2}e^{-x^2/2}. \tag{6}$$

If, in (iv), we take the same function for $f(x)$ we arrive at the function (6) again.

If we take the same function for $f(x)$ in (v), we again arrive at the function (2).

References

1. W. N. Bailey, *Some classes of functions which are their own reciprocals in the Fourier-Bessel integral transform*, Journal of the London Mathematical Society, vol. 5 (1930), pp. 258–265, (6.3).

2. ———, *Self-reciprocal functions involving confluent hyper-geometric functions*, ibid., vol. 13 (1938), pp. 111–112.

3. S. C. Dhar, *On certain functions which are self-reciprocal in the Hankel transform*, ibid., vol. 14 (1939), pp. 30–32.

4. W. T. Howell, *A note on Laguerre polynomials*, Philosophical Magazine, (7), vol. 23 (1937), pp. 807–811.

5. Brij Mohan (formerly B. M. Mehrotra), *Some theorems on self-reciprocal functions*, Proceedings of the London Mathematical Society, (2), vol. 34 (1932), pp. 231–240, §8.

6. ———, *On some self-reciprocal functions*, Bulletin of the Calcutta Mathematical Society, vol. 25 (1933), pp. 167–172, §2.

7. G. N. Watson, *Theory of Bessel Functions*, Cambridge, 1922, §§13.3, (3) and 13.3, (4).

8. E. T. Whittaker and G. N. Watson, *Modern Analysis*, Cambridge, 4th edition, 1927, §16.12.

Benares Hindu University
Benares, India

[1] Recently, Dhar [3] has given another proof of the fact that this function is R_ν. The other R_ν function $x^{\nu+2m-1/2}e^{x^2/4}W_{-3m-\nu-1/2,m}(x^2/2)$ that he gives, is the same as (5) with "$-m$" written for "m."