calculus, which can only explain its rise and growth in the sixteenth and seventeenth centuries, still has to be written.

D. J. STRUIK

Algebren. By M. Deuring. (Ergebnisse der Mathematik, vol. 4, no. 1.) *Gruppen von linearen Transformationen*. By B. L. van der Waerden (Ergebnisse der Mathematik, vol. 4, no. 2.) Berlin, Springer, 1935. 5+143 and 3+91 pp., respectively.

The theory of algebras, now about to enter the second century of its existence, constitutes today an integrating part of algebra and arithmetics. The most fundamental step in its development seems to have been the introduction of general reference fields, essentially due to Wedderburn. In order to describe approximately the degree of generality we may say that Wedderburn's theory holds at least for those fields which obey the theory of Galois. We quote from Dickson's *Linear Algebras* (1914): "Any linear associative algebra over a field F is the sum of a semisimple algebra and a nilpotent invariant subalgebra (the radical) each over F. A semisimple algebra is either simple or the direct sum of algebras over F. Any simple algebra over F is the direct product of a division algebra and a simple matrix algebra each over F."

Other results are found in Dickson's book *Algebras and their Arithmetics* (1923), which concludes with an instructive list of unsolved problems: (I) the determination of all division algebras, (II) the classification of nilpotent algebras, the discovery of relations between an algebra and its maximal nilpotent invariant subalgebra (the radical), (III) theory of non-associative algebras, and (IV) theory of ideals in the arithmetic of a division algebra and the extension to algebras of the whole theory of algebraic numbers.

Progress in the study of problems II and III has been moderate, in the sense that we have many beautiful special results but no general theory.

As to problems I and IV, our knowledge has advanced considerably, to say the least; this advance is reported in Deuring's report *Algebren* and Albert's Colloquium lectures *Structure of Algebras*. We quote both authors in saying that R. Brauer, H. Hasse, E. Noether and A. A. Albert have given the solution of problem I for algebraic number fields. From the authors who contributed to the solution of problem IV we single out (at the expense of others) H. Brandt who provided the fundamental idea that a left ideal in one maximal domain of integrity is a right ideal in another, and his pupil Eichler who did much (1936–1938) towards the solution of problem IV with respect to ideal classes and units. The reader will find an independent account of the ideal theory in the seventh section of Deuring's book.

Deuring's report opens in the spirit of modern abstract algebra with a streamlined introduction into the Wedderburn theory; not only algebras but rings are considered wherever possible. He then proceeds to the theory of matrix representations; following E. Noether he shows us what representations by matrices mean to the structure of the algebra and vice versa. This part of noncommutative algebra leads to the study of simple algebras (section VI); the mathematical concepts which permit an adequate treatment go back to Dickson and R. Brauer. Dickson invented the "cross products" as a tool of constructing normal simple algebras and in particular division algebras from extension fields Z of F.

Brauer defines the "Brauer group" of a field F in the following way: First of all, normal simple algebras are called equivalent if their division algebras are the same; that is, if they are complete matrix algebras over the same division algebra. He proves that these classes yield an abelian group if direct multiplication is chosen as composition.

The relations between cross products and Brauer group form the object of section V which deals with factor sets. Factor sets are sets of numbers in commutative fields which enter into the process of constructing non-commutative algebras by cross multiplication. We find their analogues in various places, as far distant as addition of one-digit numbers and generation of rings of operators in a Hilbert space. In the special case of cross multiplication which leads to the so-called Dickson algebras, they may be reduced to single numbers in the reference field F itself; the theory of Brauer classes is brought into the closest connection with the theory of norms of the extension Z over F. This link between algebras and arithmetic algebraic numbers, foreshadowed in papers of Dickson and Wedderburn, proved to be the key to the fundamental theorem on algebras over algebraic number fields, which is developed in section VII, on the basis of Hasse's theory of p-adic algebras. The p-adic fields are a generalization of the real and complex numbers, in the sense that they permit an analysis on the basis of a limit concept; real and complex numbers are more related to geometry, whereas the other p-adic fields (Hensel's invention) are related to the arithmetic of the congruence concept. Algebraic number fields may be imbedded in several ways into the complex or real numbers (infinite prime spots, corresponding to their conjugates) and the p-adic fields of Hensel, corresponding to their prime ideals. The division algebras over infinite p-adic fields (real or complex) are to be commutative or else quaternion systems, the Brauer group having one or two elements. Hasse determined the division algebras over the "finite" p-adic fields; they are all of the Dickson type and simple explicit constructions are given. Their Brauer groups are mapped by an explicit isomorphism on the rational numbers modulo 1. Normal simple algebras over an algebraic number field are described by their behavior under imbedding of the reference field into p-adic fields (fundamental theorem): (a) All except a finite number of p-adic imbeddings yield a full matrix algebra; therefore the rational number modulo 1 (Hasse symbol), which belongs to the prime spot and indicates the p-adic Brauer class, is 0 except for a finite number of cases. (b) If and only if all these numbers are 0 we have a complete matrix algebra; two algebras are equivalent if and only if their p-adic Brauer classes and therefore the Hasse symbols are the same. (c) The sum of the Hasse symbols is always zero (law of reciprocity). (d) If Hasse symbols are assigned with the only condition that (c) be satisfied, then there exists an algebra with these Hasse symbols. (e) Every division algebra is of the Dickson type and may be represented as a cross product belonging to a cyclotomic field.

This theorem is certainly one of which every mathematician would be proud *de porter le nom*; an inquiry into the priority situation is therefore interesting (and a little dangerous, too). The reviewer ventures the statement that the full theorem has been found by Hasse, Brauer, Noether, that the algebraic part including the "cyclic" Brauer group was developed independently by Albert and that the arithmetical side was also contained essentially in K. Hey's dissertation (1927–1929).

From the development after the appearance of Deuring's *Algebren* we shall only cite the subjects Riemann matrices, p-algebras and hypercomplex ideal theory (Albert, Eichler, Weyl, Witt and others).

The hypercomplex numbers have been in close touch with groups of linear substitutions; in fact, the preface to Dickson's *Linear Algebras* starts with this sentence: "The theory of linear associative algebras is essentially the theory of pairs of reciprocal linear groups or the theory of certain sets of matrices." It is therefore natural that B. L. van der Waerden's report on groups of linear transformations should contain also the theory of hypercomplex systems and their representations. We find there even the most general form of Brauer's factor sets, which Deuring has omitted in his presentation. This (entirely justified) omission is due to Deuring's tendency to present complete theories rather than individual theorems, the state of affairs in the theory of algebras permitting this luxury. Such an attitude is not possible with respect to groups of linear transformations; the essential reason is perhaps that algebras have been studied as representatives of a few large types, whereas groups are in many cases individuals or representatives of small families. Correspondingly, the essentials of the theory which Deuring covers are contained in comparatively few papers, whereas van der Waerden had to incorporate explicitly a host of individual investigations. If one should examine the Fortschritte der Mathematik, say since 1900, one would hardly find a paper related to the subject not mentioned in his book (excepting papers on continuous groups which were reserved for another report in the same collection).

Whatever the abstract algebraic method can do in the line of unification has been done, as in the theory of the classical linear groups (like the orthogonal groups, and so on), or the theory of representations of general classes like hypercomplex systems, finite groups, bounded representations of arbitrary groups; but in fields like groups of given degree (2, 3, 4), or the representation of individual groups (like modular groups), the special character of the problem necessitates a special treatment.

MAX ZORN

Mécanique Statistique Quantique. By Francis Perrin. (Traité du Calcul des Probabilités et de ses Applications, vol. 2, no. 5.) Paris, Gauthier-Villars, 1939. 224 pp.

This book forms a complete introductory outline of modern statistical mechanics. The first part (five chapters) deals with the rudimentary notions of classical statistical mechanics of Hamiltonian systems, ergodic theory, canonical ensembles, equipartition, coupled systems, thermostats, the thermodynamic quantities and laws, with applications to perfect gases and radiation. The second part (four chapters) introduces the rudiments of the quantum theory and extends to systems obeying its laws many of the considerations of the first part. The third part (entitled *Statistique Quantique des Systèmes Indiscernables*, eight chapters) forms the main body of the work, to which the earlier parts form a sort of introduction. The principles of indistinguishability and exclusivity are introduced and the Bose-Einstein and Fermi-Dirac schemata established. The three laws of thermodynamics are then derived from the statistical theory, and applications are made to the chemical constants, gaseous degeneres-