By Theorem 2, the solutions of the equation (17) are given by (16).

If $x_i=\rho_i$, $y_k=\sigma_k$ is any solution of (13) and we choose $\alpha_i=\rho_i$, $\mu_k=\sigma_k$, $\lambda=f(\rho)$, we have that $s=0$ and the solution becomes $x_i=\rho_i K^{n-1}$, $y_k=\sigma_k K^{n+1}$, where $K=A\lambda(AD-BC)$, which is equivalent to the given solution provided $K\neq 0$; that is, provided $x_i=\rho_i$, $y_k=\sigma_k$ is not a solution of (14). It will be noted that if $K\neq 0$, then $t\neq 0$.

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A MULTIPLE NULL-CORRESPONDENCE AND A SPACE CREMONA INVOLUTION OF ORDER $2n-1$[1]

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PART I. A NULL-SYSTEM $(1, mn, m+n)$ BETWEEN THE PLANES AND POINTS OF SPACE $(m, n=1, 2, 3, \cdots)$

1. **Introduction.** Consider a curve δ_m of order m having $m-1$ points in common with a straight line d, and a curve δ_n' of order n having $n-1$ points in common with a straight line d', $(m, n=1, 2, 3, \cdots)$. It is assumed for the present that neither δ_m nor d intersects either δ_n' or d'.

In general, through any point P of space there passes one ray ρ which intersects δ_m once and d once, and one ray ρ' which intersects δ_n' once and d' once; ρ and ρ' determine a plane π, the null-plane of P. Conversely, a plane π determines m rays ρ_i and n rays ρ_j' lying in it which intersect, a ray ρ with a ray ρ', in mn points, the null-points of the plane π.

Any point α in general position determines a ray ρ. As α describes a line l, the plane π of ρ and l contains n rays ρ', which intersect l in n points β; conversely, any point β on l determines a ray ρ' which determines with l the plane π, and π contains m rays ρ which intersect l in m points α—one being the original α. Thus an (m, n) correspondence is set up among the points of l with valence zero; there are $m+n$ coincidences and therefore $m+n$ points on any line l whose null-planes contain l.

2. **Planes whose null-points behave peculiarly.** We can obtain the last result by another method; this will yield additional information about planes whose null-points behave peculiarly.

Let a plane π turn about a line l as axis. A ruled surface will be generated by the m rays ρ_i lying in π. This surface is of order $m+1$; δ_m is a onefold curve on the surface and d is an m-fold line. Another

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ruled surface will be generated in this manner by the rays ρ_j' lying in π; its order is $n+1$, δ_n' is a onefold curve and d' is an n-fold line on this surface. The curve of intersection of these two surfaces is of order $(m+1)(n+1)$ and consists of l and a twisted curve k_{mn+m+n} of order $(m+1)(n+1)-1=mn+m+n$. *This* k_{mn+m+n} *is the locus of the null-points of all planes* π *through* l.

Since a plane π meets this in mn points outside l, k_{mn+m+n} must intersect l in $m+n$ points through each of which a ray ρ and a ray ρ' pass which are coplanar with l. Call such a point on l, P. The plane $\rho\rho'$ is the null-plane of P and has $mn-1$ null-points outside l, and it follows that plane $\rho\rho'$ is tangent to k_{mn+m+n} at P. *The null-planes of the* $m+n$ *points of intersection of* k_{mn+m+n} *with* l *are tangent planes of* k_{mn+m+n} *at these points.*

The line d, an m-fold line on the first of the two surfaces described above, intersects the second surface in $n+1$ points, which are m-fold points on the first surface. The line d' intersects the first of the two surfaces in $m+1$ points which are n-fold points on the second surface. These points all lie on k_{mn+m+n} and the $m+1$ are n-fold points of it and $n+1$ are m-fold points of it. k_{mn+m+n} *has* $m+1$ *n-fold points on* d' *and* $n+1$ *m-fold points on* d.

δ_m has no actual double points or other multiple points. It is, however, rational and has $(m-1)(m-2)/2$ apparent double points and its rank is $r=m(m-1)-(m-1)(m-2)=2(m-1)$; that is, the order of its developable surface is $2(m-1)$. Similarly, the order of the developable surface of δ_n' is $2(n-1)$. The line l will intersect $2(m-1)$ tangents of δ_m and $2(n-1)$ tangents of δ_n'. In the plane π through l and a tangent line t of the first group, two rays ρ coincide in the line which joins the point of tangency of t with the intersection of d and π. Of the mn null-points in the plane π, n lie on each of the other $m-2$ rays ρ, and $2n$ fall two and two together on the coinciding rays; *in these points* k_{mn+m+n} *is tangent to the plane of* l *and* t *and the number of these planes is* $2(m+n-2)$.

From the discussion of this section we have the following conclusions:

(1) *The planes,* m *of whose null points coincide with a point of* d, *envelope a surface of class* $n+1$; *and the planes,* n *of whose null points coincide with a point of* d, *envelope a surface of class* $m+1$.

(2) *The planes,* $2n$ *of whose null-points coincide two and two on a ray* ρ, *envelope a surface of class* $2(m-1)$, n *of the remaining null-points lying on each of the other* $m-2$ *rays* ρ; *the planes,* $2m$ *of whose null-points coincide two and two on a ray* ρ', *envelope a surface of class*

$2(n-1)$, m of the remaining null-points lying on each of the other $n-2$ rays ρ'.

Consider a plane π through l, whose intersection with d is also an intersection with δ_m. Call this common point of d and δ_m, Δ. Then the rays ρ_i lying in π will be the $m-1$ lines joining Δ to the $m-1$ points of intersection of δ_m and π, not lying on d, and the line λ joining Δ to the intersection of l and the plane of d and the tangent line to δ_m at Δ. This line λ will be the limiting position of a ray ρ as a plane revolves about l into the position of π.

In the osculating planes of δ_m and δ_n', three rays coincide. Therefore, *in the osculating planes of δ_m, $3n$ of the null-points coincide three and three on the triple ray; in the osculating planes of δ_n', $3m$ of the null-points coincide three and three on the triple ray.*

3. **Points whose null-planes behave peculiarly.** Consider a point P on d. The point P determines one ρ'. Any plane π through ρ' determines m rays ρ through P. Therefore π counts m times as null-plane of P. Conversely, for every plane through ρ' there fall m null-points together at P. The surface of class $n+1$ mentioned in §2 must have the planes π as tangent planes. *This surface is a ruled surface consisting of rays ρ' which intersect d, and conversely.* Call this surface Σ.

The surface formed by rays ρ' which intersect a general straight line l is (§2) of order $n+1$, and d intersects this surface in $n+1$ points. Thus there are n rays ρ' which intersect d and also an arbitrary line l. Therefore the surface Σ is of degree $n+1$. The line d is a onefold directrix on Σ_{n+1} and d' is an n-fold directrix; for, the n-ic cone of δ_n' projected from a point of d' will intersect d in n points. *The locus of points whose null-planes have m null-points coinciding is Σ_{n+1}.*

Similarly, the ruled surface Σ'_{m+1} of order $m+1$, consisting of rays ρ that intersect d', is the *locus of points whose null-planes have n null-points coinciding.*

Now Σ_{n+1} and Σ'_{m+1} have $mn+1$ generators in common. For the congruence of rays ρ has the characteristic $(1, m)$ and the congruence of rays ρ' has the characteristic $(1, n)$ so that, from Halphen's theorem,[2] there are $1\cdot 1+m\cdot n=mn+1$ common rays.

Since both rays ρ and ρ' through any point on one of these $mn+1$ common rays coincide, any plane through the ray can be taken as null-plane of the point. *Every plane of the pencil through any one of the $mn+1$ common rays has m null-points coinciding on d and n null-points coinciding on d'.*

[2] C. M. Jessop, *A Treatise on the Line Complex*, 1903, p. 259.

The intersection of Σ_{n+1} and Σ'_{m+1} is of degree $(n+1)(m+1)$. Since d' was shown to be an n-fold line on Σ_{n+1} and is clearly a onefold line on Σ'_{m+1}, d' therefore counts n times in the intersection of these two surfaces. Similarly d counts m times in the intersection. Each of the $mn+1$ common rays of the two congruences counts once in the intersection. The parts just enumerated have total degree $n+m+mn+1 = (n+1)(m+1)$. Therefore, *the locus of points whose null-planes have m null-points coinciding in one point and n null-points coinciding in another consists of the lines d and d' and the $mn+1$ common rays of the two congruences.*

Now consider a plane containing d; let it intersect d' in D' and δ'_n in n points N_i. *Every point of the n lines $D'N_i$ is a null-point of this plane*—similarly for planes through d'.

Let point P be on δ_m but not on d. One ρ' is determined but every line from P to d will be a ρ. Therefore, *any point of δ_m or δ'_n not also a point of d or d' has the pencil of planes through the ray of the opposite congruence as null-planes.*

Part II. A space Cremona involution of order $2n-1$ (n any integer)

4. **Definition.** Not every skew curve of order n has a secant meeting it in $n-1$ points, and some have only one such secant, but there are also skew curves of order n that have two $(n-1)$-secant lines. In such case they lie on a quadric surface and have a singly infinite system of such secants. The two selected must be two generators of the same regulus.

Consider a fixed twisted curve δ_n of order n having $n-1$ points in common with a fixed line d and $n-1$ points in common with another fixed line d'. This construction occurs when the two twisted curves δ'_n and δ_m in Part I are identical but lines d and d' remain skew to each other.

A general point P determines a unique line intersecting δ_n once, at A, and d once, at D, and a unique line intersecting δ_n once, at B, and d' once, at D'. We define P', the correspondent of P, to be the intersection of lines AD' and BD. It is an involution.

5. **Equations.** Let d be $x_1=0$, $x_2=0$, and d' be $x_3=0$, $x_4=0$, and the parametric equations of δ_n be

$$x_1 = (as + bt)\prod_1^{n-1}(t_is - s_it), \qquad x_2 = (cs + dt)\prod_1^{n-1}(t_is - s_it),$$

$$x_3 = (es + ft)\prod_n^{2n-2}(t_is - s_it), \qquad x_4 = (gs + ht)\prod_n^{2n-2}(t_is - s_it),$$

where (s_i, t_i), $(i=1, 2, \cdots, n-1)$, are values of the parameter at the $n-1$ points of δ_n on d, and for $i=n, n+1, \cdots, 2n-2$ are values of the parameter at the $n-1$ points of δ_n on d'. Then the equations of the involution are

$$x_1' = (ad - bc)\{(ah - bg)x_3 - (af - be)x_4\} \prod_1^{n-1} \alpha_i \prod_1^{n-1} \beta_i,$$

$$x_2' = (ad - bc)\{(ch - dg)x_3 - (cf - de)x_4\} \prod_1^{n-1} \alpha_i \prod_1^{n-1} \beta_i,$$

$$x_3' = (fg - eh)\{(cf - de)x_1 - (af - be)x_2\} \prod_n^{2n-2} \alpha_i \prod_n^{2n-2} \beta_i,$$

$$x_4' = (fg - eh)\{(ch - dg)x_1 - (ah - bg)x_2\} \prod_n^{2n-2} \alpha_i \prod_n^{2n-2} \beta_i,$$

where $\alpha_i \equiv (t_i d + s_i c)x_1 - (t_i b + s_i a)x_2$ and $\beta_i \equiv (t_i h + s_i g)x_3 - (t_i b + s_i e)x_4$. It is of order $2n-1$, n any integer.

6. **The fundamental system.** Line d is an $(n-1)$-fold fundamental line of simple contact. The $n-1$ fixed tangent planes through d are $\alpha_i = 0$, $(i=1, 2, \cdots, n-1)$. The line d is an F-line of the first species whose principal surface consists in the $n-1$ planes $\beta_i = 0$, $(i=1, 2, \cdots, n-1)$.

Line d' is an $(n-1)$-fold F-line of simple contact. The $n-1$ fixed tangent planes through d' are $\beta_i = 0$, $(i=n, n+1, \cdots, 2n-2)$. d' is an F-line of the first species whose P-surface is $\prod_n^{2n-2} \alpha_i = 0$.

Points Δ_i, $(i=1, 2, \cdots, n-1)$, intersections of d with δ_n whose parameters on δ_n are (s_i, t_i), and points Δ_i', $(i=n, n+1, \cdots, 2n-2)$, intersections of d' with δ_n, are isolated n-fold F-points whose P-surfaces are, respectively, the above mentioned fixed tangent planes $\alpha_i = 0$, $(i=1, 2, \cdots, n-1)$, and $\beta_i = 0$, $(i=n, n+1, \cdots, 2n-2)$.

The $(n-1)^2$ lines, each joining a Δ_i to a Δ_i', are simple F-lines without contact. They are F-lines of the second species.

The $(n-1)^2$ lines of intersection of the fixed tangent planes through d with the fixed tangent planes through d' are simple F-lines without contact. They are F-lines of the second species.

7. **Invariant locus.** Every point of the curve δ_n is invariant. Every line that intersects d, d', and δ_n, each once, goes over into itself although it is not pointwise invariant. The locus of these lines is the quadric surface on which d, d', and δ_n lie.

8. **Intersection of two homaloids.** Since they are surfaces of order

$2n-1$, two homaloids intersect in a space curve of order $(2n-1)^2$. The fixed part of this curve consists in the lines d and d', each counting $n(n-1)$ times, the $(n-1)^2$ lines joining the isolated n-fold F-points of d with those of d', each counting once, and the $(n-1)^2$ lines of intersection of the fixed tangent planes through d with those through d', each counting once. The order of this fixed part is $2n(n-1)+2(n-1)^2$.

The variable part of the curve of intersection is of order $2n-1$ and corresponds to the line of intersection of the two general planes which go over into the pair of homaloids.

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