ON THE COMPLETENESS OF A CERTAIN METRIC SPACE WITH AN APPLICATION TO BLASCHKE'S SELECTION THEOREM[1]

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1. **Introduction.** The purpose of this note is to prove that the metric space whose elements are the closed, bounded, non-null subsets of a complete metric space, and whose metric is the Hausdorff distance, is complete; and, using this result and others already known, to give a simple proof of Blaschke's selection theorem.

2. **Preliminaries.** Let K be a metric space with elements $x, y, \cdots$ and distance function $d(x, y)$. A sequence $x_1, x_2, \cdots$ in K such that $\sum_1^\infty d(x_i, x_{i+1})$ converges has been called an absolutely convergent sequence by MacNeille[2] [7, p. 192]. Every absolutely convergent sequence is a Cauchy sequence, and every Cauchy sequence contains absolutely convergent subsequences.

Let K^* be a metric space whose elements $X, Y, \cdots$ are the closed, bounded, and non-null subsets of K, and whose distance function $D(X, Y)$ is the Hausdorff distance between the sets X and Y (see Hausdorff [5, pp. 145–146] and Kuratowski [6, pp. 89–90]).

3. **The theorem.** *If K is complete, then K^* is also complete.*

Let $X_1, X_2, \cdots$ be any Cauchy sequence in K^*; without loss of generality we can assume that it is absolutely convergent. We shall define a set X and show that it is the limit of the given sequence. Let x_1 be any point in X_1, x_2 any point in X_2 such that $d(x_1, x_2) < D(X_1, X_2) + 2^{-1}$, x_3 any point in X_3 such that $d(x_2, x_3) < D(X_2, X_3) + 2^{-2}$, and so on. The existence of points $x_2, x_3, \cdots$ with the properties stated follows from the definition of the Hausdorff distance. Every point x_i in X_i is a member of a sequence $x_1, x_2, \cdots$ of the kind described. The sequence $x_1, x_2, \cdots$ is absolutely convergent and hence a Cauchy sequence; since K is complete, it has a limit x_0 in K. Let X_0 be the locus of all the points x_0 obtained as the limits of all possible sequences formed in the manner stated; let X be the closure of X_0. Then X is closed, bounded, and non-null, and X is in K^*. We shall show that $\lim X_k = X$. Let any $\epsilon > 0$ be given. Choose $n = n(\epsilon)$ so that $\sum_n^\infty [D(X_i, X_{i+1}) + 2^{-i}] < \epsilon/2$. Let $x^* \in X$, and let x_0 be the limit of a

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[2] Numbers in square brackets refer to the references at the end.

sequence $x_1, x_2, \cdots, x_k, \cdots$ and such that $d(x^*, x_0) < \epsilon/2$. Then the distance from x^* to X_k is equal to or less than

$$d(x^*, x_0) + \sum_k^\infty d(x_i, x_{i+1}) < d(x^*, x_0) + \sum_k^\infty [D(X_i, X_{i+1}) + 2^{-i}]$$
$$< \epsilon/2 + \epsilon/2 = \epsilon$$

if $k \geqq n$. Since every point x_k in X_k belongs to a sequence $x_1, x_2, \cdots$, the distance from any point x_k to X, which does not exceed the distance from x_k to the limit x_0 of the sequence $x_1, x_2, \cdots, x_k, \cdots$, is equal to or less than

$$\sum_k^\infty d(x_i, x_{i+1}) < \sum_k^\infty [D(X_i, X_{i+1}) + 2^{-i}] < \epsilon/2$$

if $k \geqq n$. From these facts it follows that $D(X_k, X) < \epsilon$ for $k \geqq n$, and hence that $\lim X_k = X$. Thus the (absolutely convergent) Cauchy sequence $X_1, X_2, \cdots$ in K^* has the limit X in K^*, and the proof of the theorem is complete.

4. **The space K^* when K is a Banach space.** The space K^* has additional properties when K is a Banach space, that is, a space which is linear, normed, and complete (see Banach [1, p. 53]). Let aX denote the set of elements ax, $x \varepsilon X$, when a is a real number; let $X+Y$ denote the set of elements $x+y$, $x \varepsilon X$ and $y \varepsilon Y$; let $C[X]$ denote the closed convex extension of X; and let $\rho(X)$ denote the diameter of X. Then K^* has, in addition to its elementary properties as a metric space, the following ones:

(4.1) $$D(aX, aY) = |a| D(X, Y) \quad \text{for every real number } a;$$

(4.2) $$D(X_1 + \cdots + X_n, Y_1 + \cdots + Y_n) \leqq D(X_1, Y_1) + \cdots + D(X_n, Y_n);$$

(4.3) $$D(C[X], C[Y]) \leqq D(X, Y);$$

(4.4) $$D(X + Y_1, X + Y_2) \leqq D(Y_1, Y_2);$$

(4.5) $$\rho(X_i) \leqq \rho(X_1 + X_2) \leqq \rho(X_1) + \rho(X_2),\ i = 1, 2;$$

(4.6) $$\rho(C[X]) = \rho(X), \quad D(C[X], 0) = D(X, 0).$$

The last two of these relations have been given by Birkhoff [2, pp. 368, 360]. The proofs of the others will be given elsewhere. It can be shown by means of examples that the inequality may hold in (4.4).

If the limit of a sequence $X_1, X_2, \cdots$ of convex sets in K^* is a set X, it follows from (4.3) that X also is convex. For

$D(X_i, C[X]) = D(C[X_i], C[X]) \leqq D(X_i, X)$; and since $D(X_i, X) \to 0$, $\lim X_i = C[X]$. But since a sequence has a unique limit, we have $C[X] = X$, and X is convex.

5. **Blaschke's selection theorem.** Let E be a closed and compact subset of a Banach space K, and let E^* denote the subset of K^* which consists of the closed, non-null subsets of E. Then both E and E^* are totally bounded, and E^* is closed and compact in K^* (see Hausdorff [**5**, pp. 107–108] and Kuratowski [**6**, p. 91]). Let E_C^* denote the subset of E^* which consists of convex sets. Since E^* is totally bounded, any infinite set of elements in $E^* \subset E^*$ contains a Cauchy sequence; since K^* is complete and E^* is closed, this sequence has a limit in E^*. By the result at the end of the last section, this limit element is itself a closed, convex set and therefore belongs to E_C^*. We have thus shown that E^* is closed and compact. This result is Blaschke's selection theorem extended to a Banach space (see Blaschke [**3**] and Bonnesen and Fenchel [**4**, p. 34]).

References

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