A NOTE ON MEASURE FUNCTIONS IN A LATTICE[1]

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We give first an equivalent statement of the measurability criterion of Carathéodory[2] which is applicable to an arbitrary lattice. We then study the closure with respect to finite and denumerable sums and products of the subset of measurable elements of a *modular* lattice. The case of regular[3] "outer measure functions" is then briefly discussed. The elements of the theory of lattices are presupposed.[4]

Let us consider a lattice L on which is defined a real-valued function $\mu(a)$. The elements $a \varepsilon L$ which satisfy

$$\mu(a + b) + \mu(ab) = \mu(a) + \mu(b) \tag{1}$$

for every $b \varepsilon L$ will be called μ-measurable. The totality of μ-measurable elements will be denoted by $L(\mu)$.

REMARK 1. *If L is a Boolean algebra and $\mu(0) = 0$, then $a \varepsilon L(\mu)$ if and only if $a \varepsilon L$ and satisfies the condition of Carathéodory,[5] that is,*

$$\mu(b) = \mu(ab) + \mu(b - ab) \tag{2}$$

for every $b \varepsilon L$. For, if $a \varepsilon L$ satisfies (1), *the equation* (1) *and*

$$\mu(a + (b - ab)) + \mu(0) = \mu(a) + \mu(b - ab)$$

yield (2). *The converse is proved by Carathéodory.*[6]

THEOREM 1. *If L is a modular lattice, then $L(\mu)$ is a sublattice of L.*

PROOF. Let $a, c \varepsilon L(\mu)$, $b \varepsilon L$. We obtain successively

$$\begin{aligned} \mu(a + (c + b)) + \mu(a(c + b)) &= \mu(a) + \mu(c + b) \\ &= \mu(a) + \mu(c) + \mu(b) - \mu(cb) \\ &= \mu(a + c) + \mu(b) + \mu(ac) - \mu(cb). \end{aligned}$$

Since $c \varepsilon L(\mu)$ we have

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[2] *Vorlesungen über Reelle Funktionen*, 2d edition, p. 246.

[3] Ibid., p. 258.

[4] See, for example, G. Birkhoff, *On the combination of subalgebras*, Proceedings of the Cambridge Philosophical Society, vol. 29 (1933), pp. 441–464; O. Ore, *On the foundations of abstract algebra* I, Annals of Mathematics, (2), vol. 36 (1935), pp. 406–437. The terminology and notation are those used by L. R. Wilcox and the author, *Metric lattices*, Annals of Mathematics, (2), vol. 40 (1939), pp. 309–327.

[5] Op. cit., p. 246.

[6] Ibid., p. 252.

$$\mu(c + a(c + b)) + \mu(ac) = \mu(c) + \mu(a(c + b)),$$
$$\mu(c + (a + c)b) + \mu(cb) = \mu(c) + \mu((a + c)b).$$

Using the modular law we see that $(a+c)(c+b)=c+a(c+b)=c+(a+c)b$. It is then clear that

$$\mu(ac) - \mu(cb) = \mu(a(c + b)) - \mu((a + c)b),$$

and (1) with a replaced by $a+c$ follows easily. Thus $a+c \in L(\mu)$. By duality, $ac \in L(\mu)$. This completes the proof.

DEFINITION 1. *If, for each increasing (decreasing) sequence $(a_i; i=1, 2, \cdots)$ of elements of $L(\mu)$ with a sum (product) $a \in L$, we have* $\lim \mu(a_i)=\mu(a)$ *as $i\to\infty$ we say that*[7] *L satisfies $B^+(\mu)$ $(B^-(\mu))$; if moreover* $\lim \mu(a_i+b)=\mu(a+b)$ *and* $\lim \mu(a_ib)=\mu(ab)$ *as $i\to\infty$ for each $b \in L$, we say that L satisfies B^+ (B^-).*

REMARK 2. *If L satisfies B^+ (B^-), then L satisfies $B^+(\mu)$ $(B^-(\mu))$. It suffices to take $b=a$ in the definition of B^+ (B^-).*

We shall assume throughout the remainder of this note that L is modular and that $\mu(a)$ is monotone increasing.

THEOREM 2. *A sufficient condition for closure of $L(\mu)$ with respect to denumerable sums (products) in L is that L satisfy B^+ (B^-). This condition is necessary if L satisfies $B^+(\mu)$ $(B^-(\mu))$.*

PROOF. To show that B^+ is sufficient, consider a sequence $(a_i; i=1, 2, \cdots)$ of elements of $L(\mu)$ with a sum $a \in L$. Define $c_i \equiv \sum(a_j; j=1, 2, \cdots, i)$. Clearly $a=\sum c_i$, and $(c_i; i=1, 2, \cdots)$ is increasing. By Theorem 1, $c_i \in L(\mu)$ for each $i=1, 2, \cdots$, and hence $\mu(c_i+b)+\mu(c_ib)=\mu(c_i)+\mu(b)$ for each $b \in L$. On taking the limit and using B^+ we see that $\mu(a+b)+\mu(ab)=\mu(a)+\mu(b)$. Thus $a \in L(\mu)$, and B^+ is sufficient. For the necessity, consider an increasing sequence $(a_i; i=1, 2, \cdots)$ of elements of $L(\mu)$ with a sum $a \in L$. For each $b \in L$ and each $i=1, 2, \cdots$, $a+b \geqq a_i+b$ and $ab \geqq a_ib$; and hence $\mu(a+b) \geqq \mu(a_i+b)$, $\mu(ab) \geqq \mu(a_ib)$. Define $\alpha \equiv \lim \mu(a_i+b)$ and $\beta \equiv \lim \mu(a_ib)$ as $i\to\infty$. Since $a_i \in L(\mu)$ we have $\mu(a_i+b)+\mu(a_ib)=\mu(a_i)+\mu(b)$. On taking the limit and using $B^+(\mu)$ and the fact that $a \in L(\mu)$ we obtain $\alpha+\beta=\mu(a)+\mu(b)$. It follows that $\alpha=\mu(a+b)$, $\beta=\mu(ab)$. Thus B^+ is necessary when L satisfies $B^+(\mu)$. The alternate reading is dual. The proof is complete.

DEFINITION 2. (1) *For each $a \in L$ we define* $\mu^+(a) \equiv \text{g.l.b.}[\mu(c); c \in L(\mu), c \geqq a]$, $\mu^-(a) \equiv \text{l.u.b.}[\mu(c); c \in L(\mu), c \leqq a]$.

[7] Cf. L. R. Wilcox and the author, op. cit., p. 317.

(2) *We say that* $\mu(a)$ *is outer (inner) regular*[8] *in case* $\mu(a)=\mu^+(a)$ $(\mu(a)=\mu^-(a))$ *for every* $a \in L$.

LEMMA 1. *If* $\mu(a)$ *is outer regular, then*

$$\mu(a+b)+\mu(ab) \leqq \mu(a)+\mu(b)$$

for every $a, b \in L$.

PROOF. Consider $a, b \in L$. For each $c, d \in L(\mu)$ for which $c \geqq a$, $d \geqq b$ we have $c+d \geqq a+b$, $cd \geqq ab$; and, by Theorem 1, $c+d$, $cd \in L(\mu)$. Consequently, since $\mu(a)$ is outer regular, $\mu(a+b)+\mu(ab) \leqq \mu(c+d)+\mu(cd)=\mu(c)+\mu(d)$. The lemma follows by applying a simple property of the greatest lower bound.

THEOREM 3. *If* L *satisfies* $B^+(\mu)$ $(B^-(\mu))$ *and* $\mu(a)$ *is outer (inner) regular, then* L *satisfies* B^+ (B^-).

PROOF. This follows from Lemma 1 and its dual by the method used in proving Theorem 2.

We now assume that L is closed with respect to denumerable sums and products.

LEMMA 2. *If* L *satisfies* B^- (B^+), *then for each* $a \in L$ *there is an element* $c \in L(\mu)$ *such that* $c \geqq a$ $(c \leqq a)$ *and* $\mu(c)=\mu^+(a)$ $(\mu(c)=\mu^-(a))$.

PROOF. This is an easy consequence of Theorem 2.

REMARK 3. *It is now clear that when* L *satisfies* B^- *and* $\mu(a)$ *is outer regular the distance function*[9] $\delta(a, b)=2\mu(a+b)-\mu(a)-\mu(b)$ *identifies each* $a \in L$ *with an element* $c \in L(\mu)$.

THEOREM 4. *If* L *satisfies* B^+ (B^-) *and* $\mu(a)$ *is outer (inner) regular, then an element* $a \in L$ *belongs to* $L(\mu)$ *if and only if* $\mu^-(a)=\mu(a)$ $(\mu^+(a)=\mu(a))$.

PROOF. Consider an element $a \in L$ for which $\mu^-(a)=\mu(a)$. By hypothesis and Lemma 2 there is an element $c \in L(\mu)$ such that $c \leqq a$ and $\mu(c)=\mu^-(a)$. Thus, for each $b \in L$, $\mu(a)+\mu(b)=\mu^-(a)+\mu(b)=\mu(c)+\mu(b)=\mu(c+b)+\mu(cb) \leqq \mu(a+b)+\mu(ab)$. Consequently, by Lemma 1, $a \in L(\mu)$. The converse is trivial. The alternate reading is dual. The proof is complete.

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[8] Cf. Carathéodory, op. cit., p. 258.

[9] See L. R. Wilcox and the author, op. cit., p. 311.