A FINITELY-CONTAINING CONNECTED SET[1]

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In a previous paper an example has been given of a set which, for every integer $n \geqq 2$, is the sum of n mutually exclusive connected subsets, but which is not the sum of *infinitely* many such subsets.[2] Here it is proposed to give an example of a connected set which, for every integer $n \geqq 2$, is the sum of n mutually exclusive *biconnected* subsets but which is not the sum of infinitely many mutually exclusive connected subsets. This example has the further property that, *for every such n, it contains n mutually exclusive connected subsets but it does not contain infinitely many such subsets*, being thus a *finitely-containing connected set.*[3] The method used will be a modification of that used by E. W. Miller to obtain a biconnected set without a dispersion point.[4] The *hypothesis of the continuum is assumed*, and use is made of the axiom of Zermelo.

The method used by Miller is dependent primarily upon showing

[1] Presented to the Society, April 15, 1939.

[2] P. M. Swingle, *Generalizations of biconnected sets*, American Journal of Mathematics, vol. 53 (1931), pp. 387–388. I call such a set a *finitely-divisible connected set.* A connected set is defined here so as to contain at least two points. The example there given consists of a connected set which is the sum of infinitely many mutually exclusive biconnected subsets, each with a dispersion point, and a limit point of these subsets which none of them contains.

[3] Loc. cit., p. 395, Problem 7. This example also solves the questions raised in Problems 4, 5, and 6, pp. 394–395. Problem 2 was answered in part in American Journal of Mathematics, vol. 54 (1932), pp. 532–535. On p. 533 it is proved for $n=2$ that E_n is the sum of m mutually exclusive biconnected subsets where m is an integer greater than n. And it is said that the proof is similar for $n>2$. For E_2 the proof depends upon constructing 3 biconnected sets, having only the origin in common. That a similar construction holds for any E_n, $(n>1)$, is seen as follows. The half cones $x_1^2+x_2^2+\cdots+x_{n-1}^2=ax_n^2$, $(x_n \geqq 0, -\infty<a<\infty)$, of E_n are each $n-1$ dimensional surfaces. As each one is composed of concentric spheres $x_1^2+x_2^2+\cdots+x_{n-1}^2=r^2$ as is also E_{n-1}, each half cone and E_{n-1} are topologically equivalent. As for $n=3$, E_{n-1} is the sum of n biconnected sets, with only the origin in common, a mathematical induction proof will show that this is true for $n>3$. For let the a's be divided into $C_{n+1,n}$ ($C_{n+1,n}$ is a binomial coefficient) mutually exclusive sets $N_1, \cdots, N_c$, each dense in their sum. Let, for each a of N_i, $(i=1, \cdots, c)$, $x_1^2+x_2^2+\cdots+x_{n-1}^2=ax_n^2$ be the sum of parts of the same n biconnected sets, where there is a total of $n+1$ such sets B_j, mutually exclusive except that they have the origin in common. Those B_j's determined by N_i will be represented by the subscripts of that combination of $1, 2, \cdots, n+1$, taken n at a time, that i of N_i represents. Then the above is seen to be true.

[4] E. W. Miller, *Concerning biconnected sets*, Fundamenta Mathematicae, vol. 29, pp. 123–133.

the existence of a widely connected subset M of an indecomposable continuum K. It is only the part of this subset M which is contained within a square Q_0 which causes M to be biconnected and it is this fact which enables us to show the existence of the desired set of this paper. We will take a countable infinity of mutually exclusive such squares plus interiors, Q, Q_1, Q_2, Q_3, $\cdots$, each containing points of K and having the relation with K that Miller's square $ABCD$ has. We will use Q_i as Miller does to show that a subset B_{ni}, $(i=1, 2, \cdots, n+1;$ $n=1, 2, 3, \cdots)$, of a set M is biconnected. And Q will be used to show that there cannot be infinitely many mutually exclusive such subsets of M.

Let V be a countable subset of K, which is dense in $K\cdot(Q_1+Q_2+Q_3+\cdots+Q)$. Let V_{ij}, $(i=1, 2, 3, \cdots; j=1, 2, \cdots, i+1)$, be a countable subset of V everywhere dense in V and such that (a) $V_{ij}\cdot V_{kl}$ is dense in V if $i\neq k$, (b) for any i the V_{ik}'s, $(k=1, 2, \cdots, i+1)$, are mutually exclusive, and (c) $V_{i1}+V_{i2}+\cdots+V_{i,i+1}=V$. For example V_{11} and V_{12} are mutually exclusive and $V_{11}+V_{12}=V$. Then V_{11} is divided into three mutually exclusive subsets, each dense in V, one for each of the sets V_{21}, V_{22}, V_{23} where V_{2j} is composed of such a set plus a similar subset of V_{12}. Each one of these three mutually exclusive subsets of V_{11} is then divided into four mutually exclusive sets, each dense in V, to obtain the parts of V_{31}, V_{32}, V_{33}, V_{34} contributed by V_{11}.

Let a division of V into infinitely many mutually exclusive subsets be U_1, U_2, $\cdots$, where each U_t, $(t=1, 2, \cdots)$, is everywhere dense in V. Either (1) there exists a region R of Q and a V_{ij} such that a U_t contains $R\cdot V_{ij}$, or (2) there does not exist such an R. If (2) is true, $V_{ij}-U_t\cdot V_{il}$ is dense in $V\cdot Q$ for each i, j, t. Consider case (1). Suppose for example that U_1 contains $R\cdot V_{32}$. Let R_1 be any region contained in R. Then U_1 contains a subset of V_{rj}, $(r>3)$, which is dense in $V_{rj}\cdot R_1$, since $V_{32}\cdot R_1$ contains such a subset because of (a) above. Hence U_t, $(t\neq 1)$, cannot contain a $V_{rj}\cdot R_1$, since U_1 and U_t are mutually exclusive. Suppose now that there exist a U_t, $(t\neq 1)$, U_2 say, which contains a $V_{3f}\cdot R_1$, $(f\neq 2$, but equals 1 say), for some R_1 of R. Hence as above U_t, $(t\neq 2)$, does not contain a $V_{rj}\cdot R_2$, where R_2 is any region of R_1. There may exist now a U_t, $(t\neq 1, 2)$, U_3 say, which contains a $V_{3f}\cdot R_2$ for $f\neq 1, 2$ but $f=3$ say. However since the U_t's are contained in $V_{31}+V_{32}+V_{33}+V_{34}$, there cannot exist a region R_3 of R_2 and a U_t, $(t\neq 1, 2, 3)$, such that U_t contains $R_3\cdot V_{3f}$, $(f\neq 1, 2, 3)$, for $R_3\cdot V_{34}$ must contain $R_3\cdot(U_4+U_5+\cdots)$. Thus in this case there exists an R_2 of R such that there are at most three U_t's which contain a $V_{ij}\cdot R_3$, where R_3 is any region of R_2. Hence there exists an R_3 of R

and a U_t, U'' say, such that for every V_{ij}, $V_{ij} - V_{ij} \cdot U''$ is dense in $V \cdot R_3$. Therefore in both cases (1) and (2) above there exists a region R'' of Q and a U_t, U'' say, such that for every V_{ij}, $V_{ij} - V_{ij} \cdot U''$ is dense in $V \cdot R''$.

The proof used by Miller to show that his widely connected set M is biconnected is dependent upon having a countable subset Δ of M and upon having a set of simple closed curves within the square $ABCD$ which have nothing in common with M except points[5] of Δ. One of these simple closed curves is taken for *each* subset of $\Delta = V$ which is dense in $V \cdot R$, where R is any region containing points of V. And the simple closed curve contains from the points of V only points from this subset of $V \cdot R$. The set of such possible subsets is c, the power of the linear continuum.

Following the method of Miller arrange in a well ordered sequence the continua C_a which separate K:

$$C_1, C_2, C_3, \cdots, C_a, \cdots, \qquad a < \Omega_c,$$

where Ω_c is the first transfinite ordinal number to correspond to the cardinal number c of the linear continuum. Let the regions of Q be well ordered as well as the possible divisions $D_1, D_2, \cdots, D_a, \cdots$ of V into infinitely many mutually exclusive subsets $U_1, U_2, \cdots$. As the power of this set of regions and the power of the set of D_a's are both c, let there be a one-to-one correspondence between each of these and the sequence $C_1, C_2, \cdots, C_a, \cdots$.

Choose for each C_a, having nothing in common with the interior of the square Q, a point set M_{ia} for each i and in each Q_i construct a simple closed curve J_{ia}, exactly as Miller does for his M, using, for each i, $Q_i \cdot V$ in place of his[6] $(ABCD) \cdot \Delta$. Thus in K, exterior to Q, we have infinitely many mutually exclusive sets, $N_1, N_2, \cdots, N_i, \cdots$ say, each exactly similar to Miller's biconnected set M, except for $K \cdot Q$. In each region R_a of Q let a simple closed curve J_a' be constructed, by a method similar to that used by Miller, so that each V_{ij} is dense in $K \cdot J_a'$. Each infinite division D_a above of V determines a U_a'' and an R_a'' of Q such that, for each i, j, $V_{ij} - V_{ij} \cdot U_a''$ is dense in $V \cdot R_a''$. In each R_a'' construct a simple closed curve J_a'' such that each V_{ij} is dense in $K \cdot J_a''$ but $J_a'' \cdot U_a'' = 0$. For each C_a separating $Q \cdot K$ choose for each V_{ij} a point or vacuous set, according to whether or not $C_a \cdot V_{ij}$ is vacuous, obtaining for each such C_a an M_{ija} of Q with the properties of Miller's M_a's. No $J_a' + J_a''$ contains a point of an M_{ija} and no two M_{ija}'s consist of the same point.

[5] E. W. Miller, loc. cit., p. 129.

[6] E. W. Miller, loc. cit., pp. 128–130.

The method used is dependent upon having chosen at any time during the process, under the hypothesis of the continuum, at most a countable infinity of points in $M\cdot(C_1+C_2+\cdots+C_a)$, where $M=N_1+N_2+\cdots+V+M_{111}+M_{121}+\cdots+M_{112}+M_{122}+\cdots$. This is true here just as it was for Miller's M_a's. As the set of composants of K is of the power of the linear continuum, new points can always be chosen for new C_a's, and each choice can be made so that no composant contains more than one point of M.

The set M is widely connected, for each C_a contains at least one point of M and no composant of K contains more than one point[7] of M. Let B_{1g}, $(g=1, 2)$, contain all of $N_g+[V_{1g}+\sum_{a=1}^{\Omega_c}M_{1ga}]\times Q$, and let in addition B_{11} contain all the rest of M, with the exception of the rest of M in Q_1, and let B_{12} contain this. Hence B_{11} and B_{12} are mutually exclusive sets whose sum is M. Each is connected, for every C_a contains a point of each. Just as Miller showed, each B_{1g} is biconnected, for suppose that B_{11}, say, is the sum of the two mutually exclusive subsets W_1 and W_2. As $W_1\cdot V$ must be dense in $Q_1\cdot V$, there exists a $J_{1a}\cdot M$ of Q_1 contained entirely in $W_1\cdot V$, according to the construction of the J_{1a}'s. As B_{11} is widely connected, this is impossible. Hence M is the sum of two mutually exclusive biconnected subsets B_{11} and B_{12}.

In a similar manner for $n>1$ it is seen that M is the sum of $n+1$ mutually exclusive biconnected subsets $B_{n1}, B_{n2}, \cdots, B_{n,n+1}$, where B_{nj} contains $N_j+[V_{nj}+\sum_a M_{nja}]\times Q$ of M and B_{n1} contains all the rest of M, except the rest of M contained in Q_1, and B_{n2} contains this.

It is seen however that M is not the sum of infinitely many mutually exclusive connected subsets $T_1, T_2, \cdots$, for every region of Q contains a J_a' and so each connected set T_i would contain a U_i dense in $V\cdot J_a'$ and so dense in $V\cdot Q$. This U_i is also dense in V because of the J_{ia}'s. Thus $T_1\cdot V, T_2\cdot V, \cdots$ is a division D_j of V into infinitely many mutually exclusive subsets $U_1, U_2, \cdots$ each dense in $V\cdot Q$. Hence one of these is a U'' which does not contain a point of some J_a''. Therefore the T_i, such that $U''=T_i\cdot V$, cannot be connected.

Thus it is seen that M is an example of a finitely-divisible connected set and similarly of a finitely-containing connected set, since each connected subset of M is widely connected.

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[7] E. W. Miller, loc. cit., p. 126, Theorem 7.