TYPICALLY-REAL FUNCTIONS WITH $a_n = 0$ FOR $n \equiv 0 \pmod 4$[1]

M. S. ROBERTSON

1. **Introduction.** Let

$$f(z) = z + \sum_{2}^{\infty} a_n z^n \tag{1.1}$$

be typically-real for $|z| < 1$; that is, $f(z)$ within this circle is regular and takes on real values when and only when z is real. In particular, if $f(z)$ is univalent for $|z| < 1$ and has real coefficients, it is also typically-real. We suppose in addition that

$$a_n = 0 \qquad \text{for } n \equiv 0 \pmod 4. \tag{1.2}$$

In this paper we obtain sharp inequalities for the coefficients a_n.

Sharp inequalities for a_n are already well known[2] with the more restrictive condition

$$a_n = 0 \qquad \text{for } n \equiv 0 \pmod 2 \tag{1.3}$$

holding. In this case $|a_n| \leqq n$ with equality occurring for the odd function $(z+z^3)(1-z^2)^{-2}$. If besides, $f(z)$ is univalent and real on the real axis, the coefficients are bounded and satisfy[3] the inequalities

$$|a_{2n-1}| + |a_{2n+1}| \leqq 2, \qquad |a_3| \leqq 1. \tag{1.4}$$

With the less restrictive condition (1.2) replacing (1.3) the author obtains the following new and sharp inequalities:

$$|a_n| + 2^{-3/2}[(n-2)|a_{2m}| + n|a_2|] \leqq n, \qquad m, n \text{ odd}, n > 1; \tag{1.5}$$

$$|a_n| + 2^{-1/2}(n-1)|a_2| \leqq n, \qquad n \text{ odd}; \tag{1.6}$$

$$|a_n| + |a_2| \leqq 2^{3/2}, \qquad |a_2| \leqq 2^{1/2}, \qquad n \text{ even}. \tag{1.7}$$

In each case the equality sign holds for the typically-real function

$$z(1 - 2^{1/2}z + z^2)^{-1} = 2^{1/2}\sum_{1}^{\infty} \sin n\pi/4 \cdot z^n.$$

Since this function is also univalent for $|z| < 1$, the inequalities above

---

[1] Presented to the Society, September 8, 1939.

[2] See W. Rogosinski, *Über positive harmonische Entwicklungen und typisch-reelle Potenzreihen*, Mathematische Zeitschrift, vol. 35 (1932), pp. 93–121.

[3] See J. Dieudonné, *Polynomes et fonctions bornées d'une variable complexe*, Annales de l'École Normale Supérieure, vol. 48 (1931), pp. 247–358.

are sharp also for the class of univalent functions with real coefficients for which (1.2) holds.

Since (1.5) may be written in the form

$$(1.8) \qquad | a_{2m} | + | a_2 | \leqq 2^{3/2}\left[1 - \limsup_{n\to\infty} | a_n/n | \right],$$

(1.7) will follow at once as well as the following theorem.

THEOREM. *If within the unit circle the typically-real function*

$$f(z) = z + \sum_{2}^{\infty} a_n z^n, \qquad a_n = 0 \textit{ for } n \equiv 0 \pmod 4,$$

has $\limsup_{n\to\infty} |a_n/n| = 1$, *then* $f(z)$ *is an odd function; that is to say,* $a_n = 0$ *for* $n \equiv 0 \pmod 2$.

In a recent paper[4] the author discussed a similar problem when $a_n = 0$ for $n \equiv 0 \pmod p$, p odd, and particularly for $p = 3$. The method used in that paper does not generalize completely to $p > 3$. Certain modifications in the method were necessary to take care of asymmetric phases which appear when $p > 3$, and these are given here for $p = 4$. The method appears to fail completely for $p > 4$.

2. **Proof of the inequalities.** Let $\mathfrak{I}f(re^{i\theta}) = v(r, \theta)$, for $r < 1$. Since $f(z)$ is typically-real for $|z| = r < 1$,

$$(2.1) \qquad \begin{aligned} v(r, \theta) &> 0 \quad \text{for} \quad 0 < \theta < \pi, & v(r, \theta) &< 0 \quad \text{for} \quad \pi < \theta < 2\pi, \\ v(r, \pi - \theta) &= - v(r, \pi + \theta), & v(r, \theta) &= - v(r, - \theta). \end{aligned}$$

In what follows we shall write $v(r, \theta)$ as simply $v(\theta)$. Since also

$$(2.2) \qquad a_n = 0 \qquad \text{for } n \equiv 0 \pmod 4,$$

it follows that

$$(2.3) \qquad f(z) + f(ze^{\pi i/2}) + f(ze^{\pi i}) + f(ze^{3\pi i/2}) \equiv 0,$$

and in particular the imaginary part of the left-hand member is zero. We write this as

$$(2.4) \qquad v(\theta) + v(\pi/2 + \theta) - v(\pi - \theta) - v(\pi/2 - \theta) \equiv 0.$$

The coefficients of $f(z)$ are given by

$$(2.5) \qquad a_n = \frac{2}{\pi r^n} \int_0^{\pi} v(\theta) \sin n\theta d\theta.$$

[4] See M. S. Robertson, *On certain power series having infinitely many zero coefficients*, Annals of Mathematics, (2), vol. 40 (1939), pp. 339–352.

Let

$$\int_0^\pi v(\theta) \sin n\theta d\theta = \int_0^{\pi/4} + \int_{\pi/4}^{\pi/2} + \int_{\pi/2}^{3\pi/4} + \int_{3\pi/4}^{\pi} = I_1 + I_2 + I_3 + I_4. \tag{2.6}$$

In I_2 let $\theta = \pi/2 - \phi$ and obtain

$$I_2 = \int_0^{\pi/4} v(\pi/2 - \phi) \sin n(\pi/2 - \phi) d\phi. \tag{2.7}$$

In I_3 let $\theta = \pi/2 + \phi$ and obtain

$$I_3 = \int_0^{\pi/4} v(\pi/2 + \phi) \sin n(\pi/2 + \phi) d\phi. \tag{2.8}$$

In I_4 let $\theta = \pi - \phi$ and obtain

$$I_4 = \int_0^{\pi/4} v(\pi - \phi) \sin n(\pi - \phi) d\phi. \tag{2.9}$$

In I_1 substitute for $v(\theta)$ the value obtained from (2.4). Combining the new forms for I_1, I_2, I_3, and I_4 we have

$$\int_0^\pi v(\phi) \sin n\phi d\phi = \int_0^{\pi/4} \{Av(\pi - \phi) + Bv(\pi/2 - \phi) + Cv(\pi/2 + \phi)\} d\phi, \tag{2.10}$$

where for brevity we write

$$\begin{aligned} A &= \sin n(\pi - \phi) + \sin n\phi = 2 \sin n\pi/2 \cos n(\pi/2 - \phi), \\ B &= \sin n(\pi/2 - \phi) + \sin n\phi = 2 \sin n\pi/4 \cos n(\pi/4 - \phi), \\ C &= \sin n(\pi/2 + \phi) - \sin n\phi = 2 \sin n\pi/4 \cos n(\pi/4 + \phi). \end{aligned} \tag{2.11}$$

Thus

$$\begin{aligned} \int_0^\pi v(\phi) \sin n\phi d\phi &= 2 \sin n\pi/2 \int_0^{\pi/4} v(\pi - \phi) \cos n(\pi/2 - \phi) d\phi \\ &\quad + 2 \sin n\pi/4 \int_0^{\pi/4} v(\pi/2 - \phi) \cos n(\pi/4 - \phi) d\phi \\ &\quad + 2 \sin n\pi/4 \int_0^{\pi/4} v(\pi/2 + \phi) \cos n(\pi/4 + \phi) d\phi \\ &= K_1 + K_2 + K_3. \end{aligned} \tag{2.12}$$

In K_1 let $\phi=\pi/2-\alpha$, in K_2 let $\phi=\pi/4-\alpha$, and in K_3 let $\phi=\alpha-\pi/4$. Then

$$\begin{aligned}\int_0^{\pi} v(\phi)\sin n\phi d\phi &= 2\sin n\pi/2\int_{\pi/4}^{\pi/2} v(\pi/2+\alpha)\cos n\alpha d\alpha \\ &\quad + 2\sin n\pi/4\int_0^{\pi/2} v(\pi/4+\alpha)\cos n\alpha d\alpha.\end{aligned} \tag{2.13}$$

Hence the formula (2.5) for the coefficients a_n may be replaced by

$$\begin{aligned}a_n = \frac{4}{\pi r^n}\Bigg[\sin n\pi/2\int_{\pi/4}^{\pi/2} v(\pi/2+\phi)\cos n\phi d\phi \\ + \sin n\pi/4\int_0^{\pi/2} v(\pi/4+\phi)\cos n\phi d\phi\Bigg].\end{aligned} \tag{2.14}$$

In particular, since $a_1=1$ we have

$$\begin{aligned}1 &= \frac{4}{\pi r}\int_{\pi/4}^{\pi/2} v(\pi/2+\phi)\cos\phi d\phi \\ &\quad + \frac{2^{3/2}}{\pi r}\int_0^{\pi/2} v(\pi/4+\phi)\cos\phi d\phi.\end{aligned} \tag{2.15}$$

For even values of $n=2k$, k odd, we have

$$a_{2k} = \frac{4(-1)^{k-1}}{\pi r^{2k}}\int_0^{\pi/2} v(\pi/4+\phi)\cos 2k\phi d\phi, \tag{2.16}$$

whence follows the inequality (to be used later)

$$\frac{4}{\pi}\int_0^{\pi/2} v(\pi/4+\phi)d\phi \geqq r^{2m}\,|\,a_{2m}\,|, \tag{2.17}$$

and in addition the equality

$$\begin{aligned}&\frac{4}{\pi}\int_0^{\pi/2} v(\pi/4+\phi)d\phi \\ &\qquad = \frac{8}{\pi}\int_0^{\pi/2} v(\phi+\pi/4)\cos^2 k\phi d\phi + (-1)^k r^{2k} a_{2k}.\end{aligned} \tag{2.18}$$

From (2.14) we have for odd values of n

$$r^n|a_n| \leqq \frac{4n}{\pi}\int_{\pi/4}^{\pi/2} v(\phi+\pi/2)\cos\phi d\phi + \frac{2^{3/2}}{\pi}\int_0^{\pi/2} v(\phi+\pi/4)d\phi. \tag{2.19}$$

With the aid of (2.18) the last inequality becomes

$$\begin{aligned} r^n|a_n| &+(-1)^{k-1}2^{-1/2}r^{2k}a_{2k} \\ &\leqq \frac{4n}{\pi}\int_{\pi/4}^{\pi/2} v(\phi+\pi/2)\cos\phi d\phi + \frac{2^{5/2}}{\pi}\int_0^{\pi/2} v(\phi+\pi/4)\cos^2 k\phi d\phi \\ &\leqq \frac{4n}{\pi}\int_{\pi/4}^{\pi/2} v(\phi+\pi/2)\cos\phi d\phi + \frac{2^{5/2}k}{\pi}\int_0^{\pi/2} v(\phi+\pi/4)\cos\phi d\phi \\ &= (n-2k)\left[\frac{4}{\pi}\int_{\pi/4}^{\pi/2} v(\phi+\pi/2)\cos\phi d\phi + \frac{2^{3/2}}{\pi}\int_0^{\pi/2} v(\phi+\pi/4)\cos\phi d\phi\right] \\ &\quad +2k\left[\frac{4}{\pi}\int_{\pi/4}^{\pi/2} v(\phi+\pi/2)\cos\phi d\phi + \frac{2^{3/2}}{\pi}\int_0^{\pi/2} v(\phi+\pi/4)\cos\phi d\phi\right] \\ &\quad -\frac{2^{3/2}(n-2k)}{\pi}\int_0^{\pi/2} v(\phi+\pi/4)\cos\phi d\phi, \end{aligned}$$

whence, on account of the equalities (2.15), (2.18) with $k=1$, and (2.17) for values of $2k<n$, we have

$$\begin{aligned} r^n|a_n| &+ (-1)^{k-1}2^{-1/2}r^{2k}a_{2k} \\ &\leqq rn - \frac{2^{3/2}(n-2k)}{\pi}\int_0^{\pi/2} v(\phi+\pi/4)\cos^2\phi d\phi \\ &= rn - \frac{2^{1/2}}{4}(n-2k)\left[\frac{4}{\pi}\int_0^{\pi/2} v(\phi+\pi/4)d\phi + r^2a_2\right] \\ &\leqq rn - (2^{1/2}/4)(n-2k)\left[r^{2m}|a_{2m}| + r^2a_2\right]. \end{aligned} \tag{2.20}$$

By considering the function $-f(-z)$, which is also typically-real, we obtain an inequality similar to this last one except that a_2 and a_{2k} have been replaced by $-a_2$ and $-a_{2k}$. Consequently, on combining both inequalities and letting r approach one we have for k and n odd

$$\begin{aligned} |a_n| &+2^{-3/2}[(n-2k)|a_{2m}| \\ &+|(n-2k)a_2+(-1)^{k-1}2a_{2k}|]\leqq n, \qquad 2k<n. \end{aligned} \tag{2.21}$$

In particular, for $k=1$ we derive for n odd

$$(2.22)\qquad | a_n | + 2^{-3/2}[(n-2) | a_{2m} | + n | a_2 |] \leqq n, \qquad n > 1.$$

If in addition $m=1$, then for n odd

$$(2.23)\qquad | a_n | + 2^{-1/2}(n-1) | a_2 | \leqq n.$$

From (2.22) on dividing by n and letting $n\to\infty$ we have

$$(2.24)\qquad \begin{aligned} &| a_{2m} | + | a_2 | \leqq 2^{3/2}\left[1 - \limsup_{n\to\infty}\left|\frac{a_n}{n}\right|\right] \leqq 2^{3/2}, \\ &| a_2 | \leqq 2^{1/2}, \qquad \limsup_{n\to\infty}\left|\frac{a_n}{n}\right| \leqq 1 - 2^{-1/2} | a_2 |. \end{aligned}$$

Though (2.22), (2.23), and (2.24) hold for m either even or odd, the interesting inequalities are for n and m both odd. In this case they are sharp, as is seen from an inspection of the coefficients of the univalent function

$$z(1 - 2^{1/2}z + z^2)^{-1} = 2^{1/2}\sum_{n=1}^{\infty} \sin n\pi/4\cdot z^n.$$

RUTGERS UNIVERSITY