(where the superscript is reduced, modulo n_i, if necessary) be a homeomorphism agreeing with T on L and sending b_i^j into b_i^{j+1} (with the same convention on the superscripts). This defines T for every p of M. It is evident that $T(M)=M$ is a pointwise periodic homeomorphism.

If we now define

$$G_i = \sum_{j=1}^{n_i} b_i^j,$$

we see that each G_i is an orbit under T, and conditions (a) and (b) of the theorem are satisfied. The proof is thus complete.

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AN ENUMERATION OF LOGICAL FUNCTIONS

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In a logical calculus of m values, abbreviated by L_m, we may deal with functions of n variables. A particular function is defined in this calculus if we assign a constant value, which may be any arbitrary one of the m possible values in L_m, as the value of that function for a particular argument. It is the purpose of this note to enumerate, among all functions of n variables in L_m: those which depend on all n variables in the argument; those which depend on just $(n-1)$ of the variables in the argument, being independent of one of them; and so on; finally those which are completely independent of all the variables in the argument.

Since each variable in the argument may assume values from $1, \cdots, m$, independently, there are m^n possible arguments, and since to each argument we may assign independently, as a functional value, any of the m values $1, \cdots, m$, there are in all m^{m^n} possible functions of n variables.

Let V_n be the total number of all functions of n variables in L_m. Then we have from the above

$$V_n = m^{m^n}. \tag{1}$$

Let U_{nk} be the number of functions of n variables which depend on exactly k of them. (It is this expression for which we are seeking an explicit evaluation.) Since k variables may be selected from n of them in just $C_{n,k}$ ways, we have the relation:

$$U_{nk} = C_{n,k}U_{kk}. \tag{2}$$

Since V_n includes all functions of n variables in L_m, we may write V_n as the sum of: all functions of n variables which depend on all of them; all functions of n variables which depend on all but one of them; and so on; that is, $V_n = \sum_{k=0}^{n} U_{nk}$, which, by the relation above, becomes

$$V_n = \sum_{k=0}^{n} C_{n,k} U_{kk}. \tag{3}$$

This leads to the following implicit recursive definition for U_{kk}:

$$V_0 = U_{00}, \tag{4_0}$$

$$V_1 = C_{1,0}U_{00} + C_{1,1}U_{11}, \tag{4_1}$$

$$V_2 = C_{2,0}U_{00} + C_{2,1}U_{11} + C_{2,2}U_{22}, \tag{4_2}$$

$$\cdots\cdots\cdots\cdots\cdots,$$

$$V_k = C_{k,0}U_{00} + C_{k,1}U_{11} + \cdots + C_{k,k}U_{kk}. \tag{4_k}$$

We may solve this system of $k+1$ equations in the $k+1$ unknowns $U_{00}, U_{11}, \cdots, U_{kk}$ by the use of Cramer's rule and some manipulation of determinants to obtain the value for U_{kk}:*

$$U_{kk} = \sum_{i=0}^{k} (-1)^i C_{k,i} V_{k-i}, \tag{4}$$

which by application of (1) gives the explicit value we want for U_{kk}:

$$U_{kk} = \sum_{i=0}^{k} (-1)^i C_{k,i} m^{m^{k-i}}. \tag{5}$$

Finally, using this result in formula (2), we have the actual enumeration formula; that is, the number of functions of n variables in L_m which depend on exactly k of them is given by the formula:

$$U_{nk} = C_{n,k} \sum_{i=0}^{k} (-1)^i C_{k,i} m^{m^{k-i}}. \tag{6}$$

REMARK. We may, in formula (4), make use of the apparent analogy with the binomial expansion. If we agree to interpret, that is, symbolize, V_i as v^i, where the i in v^i is an exponent, and write this interpretation $V_i \equiv v^i$, we have, for example, $V_0 \equiv v_0 = 1$.† Then our

* I am indebted for the actual carrying out of this solution to Dr. J. C. C. McKinsey. The solution, though elementary, is quite lengthy and may be left as an exercise for the reader.

† We write $\equiv$ to mean "symbolically equals," and $=$ to mean "equals." These are not interchangeable, since V_0 is not equal to 1, but to m.

equation (4) becomes

$$(4') \qquad U_{kk} \equiv \sum_{i=0}^{k} C_{k,i}(-1)^i v^{k-i}.$$

But this expansion on the right is simply the binomial expansion for $(v-1)^k$, so our simplified formula now is

$$(7) \qquad U_{kk} \equiv (v-1)^k.$$

For the term U_{00} we get $U_{00} \equiv (v-1)^0 = 1 = v^0 \equiv V_0$ as required in (4_0). We get corresponding definitions for the others:

$$(7_1) \qquad U_{11} = V_1 - V_0,$$

$$(7_2) \qquad U_{22} = V_2 - 2V_1 + V_0,$$

$$\cdots\cdots\cdots\cdots .$$

Each of these is made explicit by the use of formula (1). The simplified formula for U_{nk} is now

$$(8) \qquad U_{nk} \equiv C_{n,k}(v-1)^k.$$

To evaluate a particular term, expand the right side, and apply the definitions: $v^0 \equiv V_0 = m^{m^0}$, $v^i \equiv V_i = m^{m^i}$.

APPLICATION. If L_m is the two valued calculus of sentences, and if we consider truth functions of two variables $f_j(p, q)$, we have $m=2$ and $n=2$. Then $V_2 = 2^{2^2} = 16$. There are 16 possible truth functions of two variables in L_m. Of these 16 functions, there are $U_{20} = C_{2,0}2^{2^0} = 2$ functions that depend on 0 variables, that is, that are independent of both p and q ("$f(p, q)$ is always true," "$g(p, q)$ is always false"). There are $U_{21} = C_{2,1}(2^{2^1} - 2^{2^0}) = 4$ functions that depend on one variable only, that is, on p alone or on q alone, but not on both ("p is true," "p is false," "q is true," and "q is false"). There are $U_{22} = C_{2,2}(2^{2^2} - 2 \cdot 2^{2^1} + 2^{2^0}) = 10$ functions that depend on both variables (all the remaining functions in V_2, including conjunction, disjunction, implication, and so on).

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