NOTE ON CERTAIN LAGRANGE INTERPOLATION POLYNOMIALS*

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Let

$$l_k(x) \equiv l_k^{(n)}(x) \equiv \frac{\phi_n(x)}{\phi_n'(x_k)(x - x_k)}, \qquad l_k(x_k) = 1,$$

where $\phi_n(x) \equiv (x-x_1)(x-x_2) \cdots (x-x_n)$; $x_k \equiv x_{k,n} \equiv \cos\theta_k \equiv \cos\theta_{k,n}$; $0<\theta_1<\theta_2< \cdots <\theta_n<\pi$; $-1<x_k<1$, $(k=1, 2, \cdots, n)$. The polynomials $l_k(x)$ of degree $n-1$ are the fundamental polynomials of Lagrange interpolation. In this note we suppose that $\phi_n(x) \equiv \phi_n(x; \alpha, \beta)$ is the Jacobi polynomial which satisfies the following differential equation:

$$(1 - x^2)\phi_n''(x) + [\alpha - \beta - (\alpha + \beta)x]\phi_n'(x) + n(n + \alpha + \beta - 1)\phi_n(x) = 0, \qquad n = 1, 2, \cdots,$$

where α, β are positive parameters.

We develop certain bounds and limiting relations for $l_k(x)$ for special values of α, β, obtaining results similar to those of Erdös, Grünwald, and Lengyel.

It is known that ([1], pp. 17, 31, 33, 35, 62):

$$\phi_n(-x;\beta,\alpha) = (-1)^n\phi_n(x;\alpha,\beta), \ \phi_n'(x;\alpha,\beta) = n\phi_{n-1}(x;\alpha+1,\beta+1),$$

$$x_{k,n}(\beta,\alpha) = -x_{n-k+1,n}(\alpha,\beta), \ l_k^{(n)}(x;\beta,\alpha) = l_{n-k+1}^{(n)}(-x;\alpha,\beta),$$

$$\int_{-1}^{1}(1+x)^{\alpha-1}(1-x)^{\beta-1}\phi_m(x)\phi_n(x)dx = 0, \qquad m \neq n,$$

$$\phi_n(x) = \frac{(-2)^n\Gamma(n+\alpha+\beta-1)\Gamma(n+1/2)}{(\pi)^{1/2}\Gamma(2n+\alpha+\beta-1)}(\sin\lambda)^{1/2-\alpha}(\cos\lambda)^{1/2-\beta} \cdot\left\{\cos\left[(2n+\alpha+\beta-1)\lambda - \frac{\pi}{4}(2\alpha-1)\right] + O\left(\frac{1}{n}\right)\right\},$$

$$\sin^2\lambda = \frac{1+x}{2}; \ -1+\epsilon \leqq x \leqq 1-\epsilon, \ \epsilon > 0.$$

If $\alpha=\beta=1/2$, $\phi_n(x)=(1/2^{n-1})\cos n\theta$, $(x=\cos\theta)$, which is the Tschebycheff polynomial of degree n.

The following lemma due to M. Riesz [2] will be useful:

* Presented to the Society, September 7, 1939.

LEMMA. *A trigonometric polynomial of degree n assumes the maximum of its absolute value at a point whose distance from any of the roots of the trigonometric polynomial is not less than* $\pi/2n$.

Let $\alpha=\beta=3/2$. Then, we have ([1], p. 17)

$$\phi_n(x) = \frac{\sin (n+1)\theta}{2^n \sin \theta}, \qquad x = \cos\theta,$$

$$l_k(x) = \frac{(-1)^{k+1}\sin^2\theta_k \sin(n+1)\theta}{(n+1)\sin\theta(\cos\theta-\cos\theta_k)}, \qquad \theta_k = \frac{k\pi}{n+1}.$$

Since $\theta_{k+1}-\theta_k=\pi/(n+1)$, the lemma shows that the maximum of $|l_k(\cos\theta)|$ occurs between θ_{k-1} and θ_{k+1} or at $x=\pm 1$, $(k=2, 3, \cdots, n-1)$. Also, $|l_1(\cos\theta)|$ and $|l_n(\cos\theta)|$ attain their maxima at $\theta=0$ and $\theta=\pi$, respectively. We find that

$$l_1^{(1)}(x) = 1, \qquad |l_1^{(2)}(x)| = |x+1/2| \leqq 3/2,$$

$$|l_2^{(2)}(x)| = |-x+1/2| \leqq 3/2,$$

$$l_1^{(n)}(1) = l_n^{(n)}(-1) = 2\cos^2\pi/[2(n+1)],$$

$$\lim_{n\to\infty} l_1^{(n)}(1) = \lim_{n\to\infty} l_n^{(n)}(-1) = 2.$$

In the following we may assume $n\geqq 3$, $2\leqq k\leqq n-1$ and $x_k\geqq 0$. We have

$$\phi_n'(x) = -\frac{(n+1)\sin\theta\cos(n+1)\theta - \cos\theta\sin(n+1)\theta}{2^n\sin^3\theta},$$

$$\phi_n''(x) = -\frac{n(n+2)\sin^2\theta\sin(n+1)\theta + 3(n+1)\sin\theta\cos\theta\cos(n+1)\theta}{2^n\sin^5\theta}$$

$$-\frac{3\cos^2\theta\sin(n+1)\theta}{2^n\sin^5\theta}.$$

Since $\phi_n''(x)$ has opposite signs at x_k and $\cos[(2k-1)\pi/(n+1)]$, we must have $\phi_n''(\bar{x})=0$ where $\bar{x}\equiv\cos\bar{\theta}\equiv\cos(k-\epsilon)\pi/(n+1)$, where $0\leqq\epsilon<1/2$.

It follows from the differential equation that $3\bar{x}\phi_n'(\bar{x})=n(n+2)\cdot\phi_n(\bar{x})$. Since $|\phi_n(\bar{x})| = |(\bar{x}-x_k)\phi_n'(\gamma)| \geqq |(\bar{x}-x_k)\phi_n'(x_k)|$, $(x_k<\gamma<\bar{x})$, we have

$$0 \leqq \bar{x} - x_k \leqq \frac{3\bar{x}}{n(n+2)}\left|\frac{\phi_n'(\bar{x})}{\phi_n'(x_k)}\right|.$$

If

$$\max_{x_{k+1}\leqq x\leqq x_{k-1}} \left| l_k^{(n)}(x) \right| = \left| l_k^{(n)}(\mu) \right|,$$

we have

$$\max_{x_{k+1}\leqq x\leqq x_{k-1}} \left| \frac{\phi_n(x)}{\phi_n'(x_k)(x - x_k)} \right| = \left| \frac{\phi_n'(\delta)}{\phi_n'(x_k)} \right| \leqq \left| \frac{\phi_n'(\bar{x})}{\phi_n'(x_k)} \right| \equiv u,$$

$$x_k < \delta < \mu < x_{k-1}.$$

Now $(n+1)\tan\bar{\theta} > 3\pi/2$ and

$$\left| \frac{\sin (k-\epsilon)\pi}{(n+1)\tan\bar{\theta}} \right| < \frac{2}{3\pi}$$

so that

$$\begin{aligned} u &= \frac{1-x_k^2}{1-\bar{x}^2}\left[\cos (k-\epsilon)\pi - \frac{\sin (k-\epsilon)\pi}{(n+1)\tan\bar{\theta}}\right] \\ &< c\left[1 + \frac{(\bar{x}+x_k)(\bar{x}-x_k)}{1-\bar{x}^2}\right] < c\left[1 + \frac{6\bar{x}^2}{n(n+2)(1-\bar{x}^2)}\,u\right] \\ &< \frac{c}{1 - \dfrac{6c\bar{x}^2}{n(n+2)(1-\bar{x}^2)}} = \frac{c}{1 - \dfrac{6c}{n(n+2)\tan^2\bar{\theta}}} \\ &< 1.87, \end{aligned}$$

where $c = 1 + 2/3\pi$. Since $|l_k(1)| = 1 + x_k < 2$, $|l_k(-1)| = 1 - x_k < 2$, we have proved [3] the following theorem.

THEOREM 1. *If* $\alpha = \beta = 3/2$ *so that* $\phi_n(x) = \sin (n+1)\theta/2^n \sin\theta$, $(x = \cos\theta)$, *then* $|l_k^{(n)}(x)| < 2$, $(-1 \leqq x \leqq 1)$, *and* $\lim_{n\to\infty} l_1^{(n)}(1) = \lim_{n\to\infty} l_n^{(n)}(-1) = 2$. *Also,* $\lim_{n\to\infty} |l_k^{(n)}(1)| = 2$ *if, and only if,* $x_{k,n}\to 1$ *as* $n\to\infty$, *and* $\lim_{n\to\infty}[\max_{-1\leqq x\leqq 1} |l_k^{(n)}(x)|] = 2$ *if, and only if,* $x_{k,n}\to\pm 1$ *as* $n\to\infty$.

Using the same method we may prove the following similar theorems.

THEOREM 2. *If* $\alpha = 1/2$, $\beta = 3/2$ *so that*

$$\phi_n(x) = \frac{\sin (2n+1)(\theta/2)}{2^n \sin (\theta/2)}, \qquad x = \cos\theta,$$

then $|l_k^{(n)}(x)| < 2$, $(-1 \leqq x \leqq 1)$, *and* $\lim_{n\to\infty} l_1^{(n)}(1) = 2$, $\lim_{n\to\infty} l_n^{(n)}(-1) = 4/\pi$.

THEOREM 3. *If* $\alpha = 3/2, \beta = 1/2$ *so that*

$$\phi_n(x) = \frac{\cos (2n + 1)(\theta/2)}{2^n \cos (\theta/2)}, \qquad x = \cos \theta,$$

then $|l_k^{(n)}(x)| < 2$, $(-1 \leqq x \leqq 1)$, $\lim_{n\to\infty} l_1^{(n)}(1) = 4/\pi$, $\lim_{n\to\infty} l_n^{(n)}(-1) = 2$.

It should be noted that the upper bound 2 in these three theorems is the least upper bound for which the inequalities remain valid.

Let ϵ be an arbitrary, fixed positive number less than 1. We suppose that x_k is restricted to an interval I where $-1+\epsilon \leqq x_k \leqq 1-\epsilon$. Using the asymptotic formula for $\phi_n(x)$ we easily obtain the following theorem.

THEOREM 4. *For all* x *and* x_k *in* I *and* $|x - x_k| \geqq \epsilon' > 0$, $|l_k^{(n)}(x)| = O(1/n)$ *as* $n \to \infty$.

Hence, if $|l_k^{(n)}(x)|$ attains its maximum in I at $x = \mu \equiv \mu_{k,n}$, we must have $\mu - x_k \to 0$ as $n \to \infty$. At $x = \mu$, we have $(\mu - x_k)\phi_n'(\mu) - \phi_n(\mu) = 0$ so that

$$\max |l_k^{(n)}(x)| = \left| \frac{\phi_n'(\mu)}{\phi_n'(x_k)} \right| = \left| \frac{n\phi_{n-1}(\mu; \alpha + 1, \beta + 1)}{n\phi_{n-1}(x_k; \alpha + 1, \beta + 1)} \right| \to 1$$

as $n \to \infty$, provided x and x_k are in I. This proves the following theorem.

THEOREM 5. *For all* x_k *such that* $-1+\epsilon \leqq x_k \leqq 1-\epsilon$, $(\epsilon > 0)$, $\max_{-1+\epsilon \leqq x \leqq 1-\epsilon} |l_k^{(n)}(x)| \to 1$ *as* $n \to \infty$.

There is a close connection between Theorems 4 and 5 and the results of Erdös and Lengyel [**4**].

REFERENCES

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2. M. Riesz, *Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome*, Jahresbericht der deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 354–368.

3. A. Schaeffer obtained a somewhat similar proof (unpublished) for Tschebycheff polynomials. Also, see Erdös and Grünwald, *Note on an elementary problem of interpolation*, this Bulletin, vol. 44 (1938), pp. 515–518.

4. P. Erdös and B. Lengyel, *On fundamental functions of Lagrangean interpolation*, this Bulletin, vol. 44 (1938), pp. 828–834.

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