A SUFFICIENT CONDITION FOR CESÀRO SUMMABILITY*

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1. Introduction. The principal object of this paper is to establish the following theorem.

THEOREM. *The series* $\sum_{n=0}^{\infty}(-1)^n a_n$ *is exactly summable*† (C, k), $(k=1, 2, 3, \cdots)$, *to the value* $\sum_{n=0}^{k-1}\Delta^n a_0/2^{n+1}$ *provided that*

$$\Delta^i a_0 = 0, \qquad \Delta^{k-1} a_0 \neq 0, \qquad i \geqq k \geqq 1. \tag{1}$$

The convergence of the series $\sum_{n=0}^{\infty}\Delta^n a_0/2^{n+1}$ implies that the given series is summable $(E, 1)$.‡ Moreover, it is known that summability $(E, 1)$ is consistent with summability (C, k). However, neither method of summability includes the other. Thus, we may write a corollary to the stated theorem.

COROLLARY. *The class of series* $\sum_{n=0}^{\infty}(-1)^n a_n$ *for which condition* (1) *is fulfilled is summable by both the* $(E, 1)$ *and the* (C, k) *methods of summation.*

2. Lemmas. The proof of the theorem involves the following lemmas.

LEMMA 1. *If* $C_{n,k}$ *denotes the ordinary binomial coefficient, then*

$$C_{n+k-i,k} = C_{n+k+1-i,k+1} - C_{n+k-i,k+1}, \qquad k \geqq 1;\ i = 0, 1, 2, \cdots, n.$$

The simplicity of the proof of this lemma justifies its omission.

LEMMA 2. *The expression for the ith difference of a product uv in terms of differences of u alone and v alone is given by the formula*§

$$\Delta^i u_\mu v_\mu = u_{\mu+i}\Delta^i v_\mu + C_{i,1}\Delta u_{\mu+i-1}\Delta^{i-1}v_\mu + \cdots + C_{i,i}v_\mu\Delta^i u_\mu. \tag{2}$$

This formula is clearly the analogue of Leibnitz' formula for the ith

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† A series is said to be *exactly* summable (C, k) provided that it is summable (C, k) but is not summable $(C, k-1)$.

‡ This symbol denotes Euler summability of order one. The Euler transformation has been studied at considerable length particularly by E. Jacobsthal, Mathematische Zeitschrift, vol. 6 (1920), pp. 100–117, and K. Knopp, Mathematische Zeitschrift, vol. 6 (1920), pp. 118–123; vol. 15 (1922), pp. 226–253.

§ G. Wallenberg and A. Guldberg, *Theorie der Linearen Differenzengleichungen*, p. 34.

derivative of a product.

LEMMA 3. *If*

$$(3)\quad T_j^{(k)} = \sum_{i=j}^{n}(-1)^{i+j}C_{n+k+1,i+k+1}C_{i,j}2^{i-j}, \qquad j = 0, 1, 2, \cdots, k-1,$$

then

$$(4)\qquad T_j^{(k)} = \frac{(-1)^{j+n}C_{n+k+1,j}}{2^{k+1}} + \frac{C_{n+k+1,k}}{2^{j+1}} + O(n^{k-1}).$$

In order to prove this result we use the binomial expansion

$$(1-2x)^{n+k+1} = \sum_{i=0}^{n+k+1} C_{n+k+1,i}(-2x)^i.$$

Divide this equation by $j!x^{k+1}$ and then differentiate j times with respect to x. As a result of the differentiation a block of j terms vanishes on the right-hand side. We have then

$$\frac{1}{j!}D_x^j\frac{(1-2x)^{n+k+1}}{x^{k+1}} = \frac{1}{j!}D_x^j\sum_{i=0}^{k}\frac{(-2)^iC_{n+k+1,i}}{x^{k+1-i}} + \sum_{i=j+k+1}^{n+k+1}(-2)^iC_{n+k+1,i}C_{i-k-1,j}x^{i-j-k-1},$$

or

$$\frac{1}{j!}D_x^j\frac{(1-2x)^{n+k+1}}{x^{k+1}} = \frac{1}{j!}D_x^j\sum_{i=0}^{k}\frac{(-2)^iC_{n+k+1,i}}{x^{k+1-i}} + \sum_{i=j}^{n}(-2)^{i+k+1}C_{n+k+1,i+k+1}C_{i,j}x^{i-j}.$$

Now, multiply the last equation by $(-1)^{k+j+1}\,2^{-(k+j+1)}$ and evaluate for $x=1$. We obtain

$$(5)\qquad \left.\frac{(-1)^{k+j+1}}{j!2^{k+j+1}}D_x^j\frac{(1-2x)^{n+k+1}}{x^{k+1}}\right|_{x=1} = \left.\frac{(-1)^{k+j+1}}{j!2^{k+j+1}}D_x^j\sum_{i=0}^{k}\frac{(-2)^iC_{n+k+1,i}}{x^{k+1-i}}\right|_{x=1} + \sum_{i=j}^{n}(-1)^{i+j}C_{n+k+1,i+k+1}C_{i,j}2^{i-j}.$$

From equations (3) and (5) we have

$$
\tag{6}
\begin{aligned}
T_j^{(k)} = {} & \frac{(-1)^{k+j+1}}{j!2^{k+j+1}} D_x^j \frac{(1-2x)^{n+k+1}}{x^{k+1}} \Bigg|_{x=1} \\
& + \frac{(-1)^{k+j}}{j!2^{k+j+1}} D_x^j \sum_{i=0}^{k} \frac{(-2)^i C_{n+k+1,i}}{x^{k+1-i}} \Bigg|_{x=1}.
\end{aligned}
$$

Of the terms in the expansion of the first expression in the right-hand member of (6) we shall retain only the term of highest order in explicit form. Since the greatest value of j is $k-1$, the remaining terms are $O(n^{k-2})$. In the second expression of the right-hand member of (6) we shall again preserve in explicit form only the term of highest order. Regardless of the value of j the terms which remain are $O(n^{k-1})$. Accordingly, we obtain the formula (4).

3. **Proof of the theorem.** The kth Cesàro mean for the series $\sum_{n=0}^{\infty} b_n$ is given by

$$
c_n^{(k)} = \frac{S_n^{(k)}}{C_{n+k,k}} = \frac{\sum_{i=0}^{n} C_{n+k-i,k} b_i}{C_{n+k,k}}.
$$

We wish to prove that if the condition (1) of the theorem obtains, then

$$
\lim_{n\to\infty} c_n^{(k)} = \sum_{n=0}^{k-1} \frac{\Delta^n a_0}{2^{n+1}},
$$

where $b_n = (-1)^n a_n$.

We have, using Lemma 1,

$$
\begin{aligned}
S_n^{(k)} &= \sum_{i=0}^{n} C_{n+k-i,k} b_i = -\sum_{i=0}^{n} C_{n+k-i,k+1} b_i + \sum_{i=0}^{n} C_{n+k+1-i,k+1} b_i \\
&= -\sum_{i=0}^{n} C_{n+k-i,k+1} b_i + \sum_{i=-1}^{n-1} C_{n+k-i,k+1} b_{i+1}.
\end{aligned}
$$

Then

$$
\tag{7}
S_n^{(k)} = -\sum_{i=0}^{n-1} C_{n+k-i,k+1} \Delta b_i + C_{n+k+1,k+1} b_0,
$$

since $C_{k,k+1}=0$. Employing this technique on (7) we obtain

$$
S_n^{(k)} = \sum_{i=0}^{n-2} C_{n+k-i,k+2} \Delta^2 b_i + C_{n+k+1,k+1} b_0 - C_{n+k+1,k+2} \Delta b_0.
$$

After n such operations on the original expression for $S_n^{(k)}$ we get

$$S_n^{(k)} = \sum_{i=0}^{n} (-1)^i C_{n+k+1,k+i+1} \Delta^i b_0. \tag{8}$$

Formula (2) of Lemma 2 now enables us to express (8) in terms of differences of a_0. Thus,

$$\begin{aligned}\Delta^i b_0 = \Delta^i(-1)^0 a_0 &= \sum_{j=0}^{i} C_{i,j}\Delta^j(-1)^{i-j}\Delta^{i-j}a_0 \\ &= \sum_{j=0}^{i} (-1)^{i-j} C_{i,j} 2^j \Delta^{i-j} a_0 = \sum_{j=0}^{i} (-1)^j C_{i,j} 2^{i-j} \Delta^j a_0.\end{aligned}$$

Then, we may write

$$S_n^{(k)} = \sum_{i=0}^{n} (-1)^i C_{n+k+1,k+i+1} \sum_{j=0}^{i} (-1)^j C_{i,j} 2^{i-j} \Delta^j a_0. \tag{9}$$

Interchanging the order of summation in (9) we have

$$S_n^{(k)} = \sum_{j=0}^{n} \Delta^j a_0 \sum_{i=j}^{n} (-1)^{i+j} C_{n+k+1,i+k+1} C_{i,j} 2^{i-j},$$

and, recalling the definition (3), we have

$$S_n^{(k)} = \sum_{j=0}^{n} T_j^{(k)} \Delta^j a_0. \tag{10}$$

Applying the condition (1) of the theorem we may write (10) in the form

$$S_n^{(k)} = \sum_{j=0}^{k-1} T_j^{(k)} \Delta^j a_0. \tag{11}$$

From formula (4) of Lemma 3 we obtain $T_j^{(k)} = C_{n+k+1,k}/2^{j+1} + O(n^{k-1})$. Then, (11) becomes $S_n^{(k)} = C_{n+k+1,k}\sum_{j=0}^{k-1}\Delta^j a_0/2^{j+1} + O(n^{k-1})$, whence

$$c_n^{(k)} = \frac{S_n^{(k)}}{C_{n+k,k}} = \frac{n+k+1}{n+1}\sum_{j=0}^{k-1}\frac{\Delta^j a_0}{2^{j+1}} + O(1/n).$$

Thus, we obtain our main result:

$$\lim_{n\to\infty} c_n^{(k)} = \sum_{j=0}^{k-1} \frac{\Delta^j a_0}{2^{j+1}}.$$

In order to complete the proof of our theorem it remains to prove that $\lim_{n\to\infty} c_n^{(k-1)}$ does not exist. We have $S_n^{(k-1)} = \sum_{j=0}^{k-1} T_j^{(k-1)} \Delta^j a_0$.

Using Lemma 3 we have

$$
\begin{aligned}
S_n^{(k-1)} &= \sum_{j=0}^{k-1} \frac{(-1)^{j+n} C_{n+k,j}}{2^k} \Delta^j a_0 + C_{n+k,k-1} \sum_{j=0}^{k-1} \frac{\Delta^j a_0}{2^{j+1}} + O(n^{k-2}) \\
&= \frac{(-1)^{k+n-1} C_{n+k,k-1}}{2^k} \Delta^{k-1} a_0 + C_{n+k,k-1} \sum_{j=0}^{k-1} \frac{\Delta^j a_0}{2^{j+1}} + O(n^{k-2}).
\end{aligned}
$$

Then

$$
\lim_{n\to\infty} c_n^{(k-1)} = \lim_{n\to\infty} \frac{S_n^{(k-1)}}{C_{n+k-1,k-1}} = \frac{\Delta^{k-1} a_0}{2^k} \lim_{n\to\infty} (-1)^{k+n-1} + \sum_{j=0}^{k-1} \frac{\Delta^j a_0}{2^{j+1}}.
$$

This limit fails to exist. Consequently, under the hypotheses of our theorem, the series $\sum_{n=0}^{\infty}(-1)^n a_n$ is not summable $(C, k-1)$.

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A GENERAL CONTINUED FRACTION EXPANSION*

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Introduction. Considerable attention has been given at various times by many writers to the function-theoretic character of continued fractions of the form

$$
1 + \frac{a_1 x}{1} + \frac{a_2 x}{1} + \cdots .
$$

Only a very restricted class of power series, the "seminormal" ones, admit an expansion into a continued fraction of this type (cf. Perron [3, p. 301]). For example, the power series expansion about the origin of the function $1+x^2$ fails to be seminormal. In §1 of this paper we show that *every* power series admits an expansion into a continued fraction of a form which is a generalization of that above. Many of the older theorems have immediate generalizations. These are presented without proof when the demonstration parallels that for the seminormal case.

In §2 we discuss the question of gaps in seminormal power series. In §3 an important special case is considered.

1. Expansions in continued fractions. Let

$$
1 + \frac{a_1 x^{\alpha_1}}{1} + \frac{a_2 x^{\alpha_2}}{1} + \cdots \tag{1.1}
$$

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