REMARKS ON THE CLASSICAL INVERSION FORMULA FOR THE LAPLACE INTEGRAL

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If a function $f(s)=f(\sigma+i\tau)$ is defined for $\sigma>0$ by the Laplace integral

$$f(s) = \int_0^\infty e^{-st}\phi(t)dt, \tag{1}$$

then the classical inversion formula is

$$\phi(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)e^{st}ds, \qquad c > 0, t > 0. \tag{2}$$

Conditions for the validity of this formula have frequently been discussed. However, the authors know of no adequate treatment* of the case when $\phi(t)$ belongs to L^2 in $(0, \infty)$:

$$\int_0^\infty |\phi(t)|^2 dt < \infty. \tag{3}$$

We employ here the usual notation,

$$\underset{a\to\infty}{\text{l.i.m.}}\ \phi_a(t) = \phi(t),$$

to mean that $\phi_a(t)$ and $\phi(t)$ belong to L^2 in $(-\infty, \infty)$ and that

$$\lim_{a\to\infty}\int_{-\infty}^\infty |\phi_a(t) - \phi(t)|^2 dt = 0.$$

It is clear first that if (3) holds then (1) converges absolutely for $\sigma>0$, since

$$|f(\sigma + i\tau)|^2 = \left|\int_0^\infty e^{-st}\phi(t)dt\right|^2 \leqq \int_0^\infty e^{-2\sigma t}dt \int_0^\infty |\phi(t)|^2 dt.$$

Moreover, by the Plancherel theorem regarding Fourier transforms,

$$\underset{a\to\infty}{\text{l.i.m.}} \int_0^a e^{-i\tau t}\phi(t)dt$$

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* But compare G. Doetsch, *Bedingungen für die Darstellbarkeit einer Funktion als Laplace Integral und eine Umkehrformel für die Laplace-Transformation*, Mathematische Zeitschrift, vol. 42 (1936), p. 272, Theorem 1.

exists. We denote it by $f(i\tau)$. The same theorem gives us at once that

$$\underset{a\to\infty}{\text{l.i.m.}}\ \frac{1}{2\pi}\int_{-a}^{a} f(i\tau)e^{i\tau t}d\tau = \begin{cases}\phi(t), & t>0,\\ 0, & t<0,\end{cases}$$

or

$$\underset{a\to\infty}{\text{l.i.m.}}\ \frac{1}{2\pi i}\int_{-ia}^{ia} f(s)e^{st}ds = \begin{cases}\phi(t), & t>0,\\ 0, & t<0.\end{cases} \tag{4}$$

Hence (2) with $c=0$ is valid in the sense of (4). However, if $c>0$, (2) is no longer valid in this sense.

If $c>0$, it is again clear from the Plancherel theorem that

$$\underset{a\to\infty}{\text{l.i.m.}}\ \frac{1}{2\pi}\int_{-a}^{a} f(c+i\tau)e^{i\tau t}d\tau = \begin{cases}e^{-ct}\phi(t), & t>0,\\ 0, & t<0.\end{cases}$$

But this does not imply that

$$\underset{a\to\infty}{\text{l.i.m.}}\ \int_{c-ia}^{c+ia} f(s)e^{st}ds = \begin{cases}\phi(t), & t>0,\\ 0, & t<0,\end{cases}$$

unless $f(s)$ is identically zero. For, set $\phi_a(t)=\int_{-a}^{a} f(c+i\tau)e^{i\tau t}d\tau$. It will be sufficient to show that

$$\int_{-\infty}^{\infty} e^{2ct}\,|\,\phi_a(t)\,|^2 dt = \infty \tag{5}$$

for some a. Choose a so that

$$f(c+ia) \neq f(c-ia). \tag{6}$$

This is possible, for otherwise we should have by use of (1) that

$$\int_0^{\infty} e^{-ct}\phi(t)\sin at\,dt = 0$$

for all a. By the uniqueness theorem for the Fourier sine transform this would imply that $\phi(t)$ is equivalent to zero and that $f(s)$ is identically zero. In fact we see that (6) may be satisfied for some a in every interval however small.

An integration by parts of the integral defining $\phi_a(t)$ gives

$$\begin{aligned}\phi_a(t) &= \frac{f(c+ia)e^{iat}-f(c-ia)e^{-iat}}{it} - \frac{1}{t}\int_{-a}^{a} e^{i\tau t}f'(c+i\tau)d\tau\\ &= \frac{f(c+ia)e^{iat}-f(c-ia)e^{-iat}}{it} + o\left(\frac{1}{|t|}\right), \qquad |t|\to\infty,\end{aligned}$$

$$= \frac{e^{iat}}{it}\left[\{f(c+ia) - f(c-ia)\} + f(c-ia)(1 - e^{-2iat})\right]$$
$$+ o\left(\frac{1}{|t|}\right).$$

Hence

$$t^2 |\phi_a(t)|^2 \geqq \left[\frac{k}{2} - 2l\,|\sin at|\right]^2, \qquad t > t_0,$$

where t_0 is a sufficiently large positive number and

$$k = |f(c+ia) - f(c-ia)| \neq 0, \qquad l = |f(c-ia)|.$$

Since $2l|\sin at| < k/4$ in an interval of length δ, say, about $t=0$ and in intervals congruent to this one, modulo π/a, it is clear that the integrand of (5) exceeds $k^2e^{2ct}/16t^2$ in infinitely many intervals of length δ. This is sufficient to insure the divergence of the integral.

We collect our results in the following form:

THEOREM. *If $\phi(t)$ belongs to L^2 in $(0, \infty)$, then it has a Laplace transform $f(\sigma+i\tau)$ defined for $\sigma>0$ by the absolutely convergent integral*

$$f(\sigma + i\tau) = \int_0^\infty e^{-(\sigma+i\tau)t}\phi(t)dt,$$

and for $\sigma=0$ by

$$f(i\tau) = \underset{a\to\infty}{\text{l.i.m.}} \int_0^a e^{-i\tau t}\phi(t)dt.$$

The inversion formula

$$\underset{a\to\infty}{\text{l.i.m.}} \frac{1}{2\pi i}\int_{c-ia}^{c+ia} f(s)e^{st}ds = \begin{cases}\phi(t), & t>0,\\ 0, & t<0,\end{cases}$$

is false ($f(s)\not\equiv 0$) for $c>0$ and valid for $c=0$. For all $c \geqq 0$

$$\underset{a\to\infty}{\text{l.i.m.}} \frac{1}{2\pi}\int_{-a}^{a} f(c+i\tau)e^{i\tau t}d\tau = \begin{cases}e^{-ct}\phi(t), & t>0,\\ 0, & t<0.\end{cases}$$

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