SOME THEOREMS ON SUBSEQUENCES†

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It is obvious that, for any real sequence for which the sum Σ of the moduli of its elements exists and is finite, there exists a subsequence such that the modulus of the sum of its elements is not less than $\Sigma/2$. The purpose of this paper is to formulate and investigate analogous statements for complex sequences.

Let $\mathfrak{A}$ be the class of sequences, finite or infinite, $\{a_k\}$ (denoted alternatively by A) of non-zero complex numbers for which $\sum |a_k| < \infty$, and $\{a_j'\}$ (denoted alternatively by S), the general subsequence of $\{a_k\}$ for fixed $\{a_k\}$. Let $\mathfrak{B}$ be the class of sequences $\{b_k\}$ (denoted alternatively by B) of non-zero complex numbers for which $\sum |b_k| = \infty$, and $\{b_j'\}$ (denoted alternatively by T), the general subsequence of $\{b_k\}$ for fixed $\{b_k\}$.

The following facts will be established: (i) Given any sequence $\{a_k\}\epsilon\mathfrak{A}$, there then exists a subsequence $\{a_j^*\}$ for which $|\sum a_j^*| = \sup_S |\sum a_j'|$. (ii) If $\rho \equiv \inf_A \max_S |\sum a_j'| / \sum |a_k|$, then $\rho = 1/\pi$. (iii) No sequence $\{a_k\}\epsilon\mathfrak{A}$ exists for which $\max_S |\sum a_j'| / \sum |a_k| = \rho$. (iv) Given any sequence $\{b_k\}\epsilon\mathfrak{B}$, there exists a subsequence $\{b_j^*\}$ such that‡

$$\limsup |\textstyle\sum' b_j^*| / \sum_1^N |b_k| = \sup_T \limsup_N |\sum' b_j'| / \sum_1^N |b_k|$$

$$= \limsup_N \sup_T |\textstyle\sum' b_j'| / \sum_1^N |b_k| = \limsup_N \max_T |\sum' b_j'| / \sum_1^N |b_k|.$$

(v) If $\sigma = \inf_B \max_T \limsup_N |\sum' b_j'| / \sum_1^N |b_k|$, then $\sigma = \rho$. (vi) There exists a sequence $\{b_k\}\epsilon\mathfrak{B}$ for which $\max_T \limsup_N |\sum' b_j'| / \sum_1^N |b_k| = \sigma$.

Use will be made of abbreviations of the following sort: $A_k \equiv |a_k|$, $\phi_k \equiv \arg a_k$. For definiteness, the function "arg" will mean, throughout this paper, principal argument. Given any sequence $\{a_k\}\epsilon\mathfrak{A}$, define

$$F(\phi) \equiv \sum_{\cos(\phi-\phi_k)>0} A_k \cos(\phi - \phi_k)$$
$$= \sum A_k\{\cos(\phi - \phi_k) + |\cos(\phi - \phi_k)|\}/2, \qquad 0 \leqq \phi \leqq 2\pi.$$

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‡ The notation $\sum'$ indicates summation over precisely those elements of the subsequence which occur among the elements of the original sequence summed elsewhere in the formula.

Being continuous, $F(\phi)$ attains its supremum. In what follows, to and including Theorem 3, $\{a_k\}$ will signify an arbitrary but fixed sequence of class $\mathfrak{A}$.

THEOREM 1. *Let* ϕ^* *be such that* $F(\phi^*)=\max F(\phi)$, *and let* $\{a_j^*\}$ *be the sequence of those elements of* $\{a_k\}$ *for which* $\cos(\phi^*-\phi_k)>0$. *Then* $\sup_S |\sum a_j'| = F(\phi^*) = |\sum a_j^*|$.

PROOF. Let $\{a_j'\}$ be any subsequence of $\{a_k\}$, and define $\phi = \arg \sum a_j'$. Then

$$\begin{aligned} \left| \sum a_j^* \right| &\geqq \sum A_j^* \cos(\phi^* - \phi_j^*) = F(\phi^*) \geqq F(\phi) \\ &= \sum_{\cos(\phi-\phi_k)>0} A_k \cos(\phi - \phi_k) \geqq \sum A_j' \cos(\phi - \phi_j') = \left| \sum a_j' \right|. \end{aligned}$$

This establishes (i).

COROLLARY 1.1. *In the notation of Theorem* 1, $\phi^* = \arg \sum a_j^*$.

PROOF. Taking $\{a_j'\} \equiv \{a_j^*\}$ in the inequalities of Theorem 1, we see that $|\sum a_j^*| = \sum A_j^* \cos(\phi^* - \phi_j^*)$. That is, the modulus of $\sum a_j^*$ is equal to that of its projection on the ray of angle ϕ^*.

The following theorem and its corollary provide a sort of converse or dual of Theorem 1 and Corollary 1.1:

THEOREM 2. *Let* $\{\bar{a}_j\}$ *be a subsequence of* $\{a_k\}$ *for which* $|\sum \bar{a}_j| = \max_S |\sum a_j'|$, *and let* $\bar{\phi} \equiv \arg \sum \bar{a}_j$. *Then* $\max F(\phi) = |\sum \bar{a}_j| = F(\bar{\phi})$.

PROOF. Let ϕ be any angle, $(0 \leqq \phi \leqq 2\pi)$, and $\{a_j'\}$ the sequence of those elements of $\{a_k\}$ for which $\cos(\phi - \phi_k) > 0$. Then

$$\begin{aligned} F(\bar{\phi}) &= \sum_{\cos(\bar{\phi}-\phi_k)>0} A_k \cos(\bar{\phi} - \phi_k) \geqq \sum \bar{A}_j \cos(\bar{\phi} - \bar{\phi}_j) \\ &= \left| \sum \bar{a}_j \right| \geqq \left| \sum a_j' \right| \geqq \sum_{\cos(\phi-\phi_k)>0} A_k \cos(\phi - \phi_k) = F(\phi). \end{aligned}$$

COROLLARY 2.1. *In the notation of Theorem* 2, $\{\bar{a}_j\}$ *is the sequence of those elements of* $\{a_k\}$ *for which* $\cos(\bar{\phi} - \phi_k) > 0$.

PROOF. Taking $\phi = \bar{\phi}$ in the inequalities of Theorem 2, we see that

$$\sum_{\cos(\bar{\phi}-\phi_k)>0} A_k \cos(\bar{\phi} - \phi_k) = \sum \bar{A}_j \cos(\bar{\phi} - \bar{\phi}_j).$$

In conjunction with Theorem 3 (below), this proves the assertion.

THEOREM 3. *In the notation of Theorem* 2, *there exists no element* a_κ *of* $\{a_\kappa\}$ *for which* $\cos(\bar{\phi} - \phi_\kappa) = 0$.

PROOF. If there were such an element, then $|\sum \bar{a}_j \pm a_\kappa| > |\sum \bar{a}_j|$, so that addition of a_κ to $\{\bar{a}_j\}$, if it were not already therein contained, or removal of it, if it were, would provide a subsequence of $\{a_k\}$ to establish that $|\sum \bar{a}_j| < \max_S |\sum a_j'|$, contrary to the definition of $\{\bar{a}_j\}$.

THEOREM 4. $\rho = 1/\pi$.

PROOF. First,

$$\int_0^{2\pi} F(\phi)d\phi = 2\sum A_k.$$

Thus $\max F(\phi) \geqq \sum A_k/\pi$, whence, by Theorem 1, $\rho \geqq 1/\pi$. To show that $\rho \leqq 1/\pi$, consider the sequence over ν of particular finite sequences $\{{}_\nu a_k\}$, where ${}_\nu a_k \equiv \exp\{k\pi i/(2\nu+1)\}$, $(k = -2\nu, -2\nu+1, \cdots, 0, 1, \cdots, 2\nu, 2\nu+1)$. By Corollary 2.1, for given ν any subsequence $\{{}_\nu a_j'\}$ of $\{{}_\nu a_k\}$ the sum of whose elements is of maximum modulus consists of those elements whose arguments lie in a certain sector of aperture π. By the symmetry of the sequence $\{{}_\nu a_k\}$, the midray of such a sector must lie either on a vector ${}_\nu a_k$ or midway between two such vectors which are adjacent. In the latter case, however, Theorem 3 would be violated. Hence the former must obtain, and thus those elements of $\{{}_\nu a_k\}$ for which $-\pi/2 < k\pi/(2\nu+1) < \pi/2$ constitute a subsequence the sum of whose elements is of maximum modulus. Hence, if $S(\nu)$ denotes the general subsequence $\{{}_\nu a_j'\}$ of $\{{}_\nu a_k\}$,

$$\max_{S(\nu)} \left| \sum_j {}_\nu a_j' \right| / \sum_k {}_\nu A_k = \sum_{k=-\nu}^{\nu} \cos\{k\pi/(2\nu+1)\}/\{2(2\nu+1)\}$$
$$= 1/\{2(2\nu+1)\sin[\pi/\{2(2\nu+1)\}]\};$$

and, as $\nu \to \infty$, this tends monotonely to $1/\pi$. This establishes (ii).

THEOREM 5. *There exists no sequence* $\{a_k\} \epsilon \mathfrak{A}$ *for which* $F(\phi)$ *is constant.*

PROOF. If there were such a sequence $\{a_k\}$ then, by Theorem 1, for each ϕ the sequence $\{a_j^*\}$ of those elements of $\{a_k\}$ for which $\cos(\phi - \phi_k) > 0$ would be such that $|\sum a_j^*| = \max_S |\sum a_j'|$. Hence, by Corollary 1.1 and Theorem 3, there would exist no non-zero element of $\{a_k\}$, contrary to the definition of $\mathfrak{A}$.

THEOREM 6. *Given an arbitrary sequence, finite or infinite, of pairs* (C_k, ψ_k), *where the* ψ_k *are real numbers and the* C_k *positive numbers with* $\sum C_k < \infty$, *then* $\Phi(\phi) \equiv \sum C_k |\cos(\phi - \psi_k)|$ *is not constant.*

PROOF. The sequence $\{a_k\}$ defined thus: $a_{2k-1} \equiv C_k \exp(i\psi_k)$, $a_{2k} \equiv C_k \exp[i(\psi_k - \pi)]$, is of class $\mathfrak{A}$, and

$$F(\phi) = \sum_{\cos(\phi-\psi_k)>0} C_k \cos(\phi - \psi_k) + \sum_{\cos(\phi-\psi_k)<0} C_k \cos(\phi + \pi - \psi_k)$$
$$= \sum C_k \left| \cos(\phi - \psi_k) \right| = \Phi(\phi).$$

The conclusion now follows from Theorem 5.

THEOREM 7. *There exists no sequence* $\{a_k\} \epsilon \mathfrak{A}$ *for which it is true that* $\max_S \left|\sum a_j'\right| / \sum A_k = \rho$.

PROOF. If there were such a sequence $\{a_k\}$, then, by Theorem 1, $F(\phi) \leqq \rho \sum A_k$ for all ϕ. Hence

$$\int_0^{2\pi} \left| \rho \sum A_k - F(\phi) \right| d\phi = \int_0^{2\pi} \left\{ \rho \sum A_k - F(\phi) \right\} d\phi$$
$$= 2 \sum A_k - 2 \sum A_k = 0.$$

By continuity, then, $F(\phi) = \rho \sum A_k$ for all ϕ. But by Theorem 5 this is impossible. This establishes (iii).

LEMMA 8.1. *Let X be an aggregate of elements x of any sort, and $\{f_N\}$ any sequence of functionals over X. Then* $\sup_x \limsup_N f_N(x) \leqq \limsup_N \sup_x f_N(x)$.

PROOF. For each N and for all x, $f_N(x) \leqq \sup_x f_N(x)$. Hence, for all x, $\limsup_N f_N(x) \leqq \limsup_N \sup_x f_N(x)$, and the conclusion follows.

REMARK. Equality in the conclusion of Lemma 8.1 is not implied by the hypotheses. For, if we let X represent the totality of real numbers and define $f_N(1/N) = 1$, $f_N(x) = 0$ for $x \neq 1/N$, $(N = 1, 2, \cdots)$, it follows that $\limsup_N f_N(x) = 0$ for each x, so that $\sup_x \limsup_N f_N(x) = 0$, whereas $\sup_x f_N(x) = 1$ for each N, so that $\limsup_N \sup_x f_N(x) = 1$.

THEOREM 8. *Let* $\{b_k\} \epsilon \mathfrak{B}$ *be arbitrary. Then there exists a subsequence* $\{b_j^*\}$ *of* $\{b_k\}$ *for which*

$$\limsup_N \left| \sum{}' b_j^* \right| / \sum_1^N B_k = \sup_T \limsup_N \left| \sum{}' b_j' \right| / \sum_1^N B_k$$
$$= \limsup_N \sup_T \left| \sum{}' b_j' \right| / \sum_1^N B_k = \limsup_N \max_T \left| \sum{}' b_j' \right| / \sum_1^N B_k.$$

PROOF. By Theorem 1, for each N there exists a subsequence $\{b_j^{(N)}\}$ of $\{b_k\}$ for which $\left|\sum{}' b_j^{(N)}\right| / \sum_1^N B_k = \sup_T \left|\sum{}' b_j'\right| / \sum_1^N B_k$. Let $\{N(\nu)\}$, $(\nu = 1, 2, \cdots)$, be a subsequence of $\{N\}$ such that

$$\lim \left| \sum_j{}' {}_\nu b_j \right| \Big/ \sum_1^{N(\nu)} B_k = \limsup_N \left| \sum_j{}' b_j^{(N)} \right| \Big/ \sum_1^N B_k,$$

and such that

$$\sum_1^{N(\nu-1)} B_k \Big/ \sum_1^{N(\nu)} B_k < 1/2^{\nu+1},$$

where the notation ${}_\nu b_j$ represents $b_j^{(N)}$ with $N = N(\nu)$. Define the subsequence $\{b_j^*\}$ of $\{b_k\}$ in such a manner that its elements coincide in order with those of $\{{}_\nu b_j\}$ in the subscript interval (with respect to the original sequence $\{b_k\}$) $N(\nu-1) < k \leqq N(\nu)$ for all ν, $(N(0) \equiv 0)$. Now

$$\limsup_N \left| \sum{}' b_j^* \right| \Big/ \sum_1^N B_k \geqq \limsup_\nu \left| \sum{}' b_j^* \right| \Big/ \sum_1^{N(\nu)} B_k,$$

so that from the inequality

$$\left| \sum{}' b_j^* \right| \Big/ \sum_1^{N(\nu)} B_k \geqq \left| \sum_j{}' {}_\nu b_j \right| \Big/ \sum_1^{N(\nu)} B_k - 1/2^\nu,$$

it follows that

$$\limsup_N \left| \sum{}' b_j^* \right| \Big/ \sum_1^N B_k \geqq \limsup_N \max_T \left| \sum{}' b_j' \right| \Big/ \sum_1^N B_k.$$

But that

$$\limsup_N \left| \sum{}' b_j^* \right| \Big/ \sum_1^N B_k \leqq \sup_T \limsup_N \left| \sum{}' b_j' \right| \Big/ \sum_1^N B_k$$

is obvious, and that

$$\sup_T \limsup_N \left| \sum{}' b_j' \right| \Big/ \sum_1^N B_k \leqq \limsup_N \max_T \left| \sum{}' b_j' \right| \Big/ \sum_1^N B_k$$

follows from Lemma 8.1. The conclusion follows. This establishes (iv).

LEMMA 9.1. $\sigma \geqq \rho$.

PROOF. By Theorem 8,

$$\begin{aligned} \sigma &\equiv \inf_B \max_T \limsup_N \left| \sum{}' b_j' \right| \Big/ \sum_1^N B_k \\ &= \inf_B \limsup_N \max_T \left| \sum{}' b_j' \right| \Big/ \sum_1^N B_k \geqq \rho, \end{aligned}$$

which establishes the lemma.

Consider now the sequence $\{b_k^*\}$ defined thus: $b_k^* \equiv e^{ik}$, $(k=1, 2, \cdots)$, and define

$$F_N(\phi) \equiv \sum_1^N \{\cos(\phi - k) + |\cos(\phi - k)|\}/2N, \qquad 0 \leqq \phi \leqq 2\pi.$$

LEMMA 9.2. $\text{Lim}_N \text{ osc}_\phi F_N(\phi) = 0$.

PROOF. Let $\epsilon > 0$ be arbitrary; let K be such that, for each ϕ, $\phi \equiv p(\phi) + \eta_\phi \pmod{2\pi}$ for some η_ϕ for which $|\eta_\phi| < \epsilon$ and some integer $p(\phi)$ for which $0 \leqq p(\phi) \leqq K$; and let N be such that $K/N < \epsilon$. Then, for each ϕ,

$$\begin{aligned} |F_N(\phi) - F_N(0)| \leqq \Bigg| \sum_{k=1}^{N-p(\phi)} &\{\cos(k - \eta_\phi) \\ &+ |\cos(k - \eta_\phi) - \cos k - |\cos k|\}\Bigg| /2N \\ &+ \Bigg| \sum_{k=1}^{p(\phi)} \{\cos(\phi - k) + |\cos(\phi - k)|\}\Bigg| /2N \\ &+ \Bigg| \sum_{N-p(\phi)+1}^{N} \{\cos k + |\cos k|\}\Bigg| /2N < 3\epsilon. \end{aligned}$$

This establishes the lemma.

LEMMA 9.3. $\text{Lim}_N F_N(\phi) = \rho$ *uniformly in* ϕ.

PROOF. The assertion follows from Lemma 9.2 and the fact that, for each N, $\int_0^{2\pi} F_N(\phi) d\phi = 2$.

THEOREM 9. $\sigma = \rho$.

PROOF. Applying Theorem 2 to the (finite) sequence of those elements of $\{b_k^*\}$ for which $k \leqq N$, we find that

$$\max_T \left| \sum{}' b_j^{*\prime} \right| / \sum_1^N B_k^* = \max_\phi F_N(\phi),$$

which tends to ρ, by Lemma 9.3. By Theorem 8 and Lemma 9.1, this establishes the theorem, and hence also (v).

THEOREM 10. *There exist an uncountably infinite number of subsequences* $\{\bar{b}_j^*\}$ of $\{b_k^*\}$ *for which*

$$\lim_N \left| \sum{}' \bar{b}_j^* \right| / \sum_1^N B_k^* = \max_T \limsup_N \left| \sum{}' b_j^{*\prime} \right| / \sum_1^N B_k^* = \rho = \sigma.$$

PROOF. Let ϕ' be arbitrary, and let $\{\bar{b}_j^*\}$ be the sequence of those elements of $\{b_k^*\}$ for which $\cos(\phi' - \phi_k^*) > 0$. Then, by inequalities like those used in the proof of Theorem 2, for each N,

$$F_N(\phi') \leqq \left| \sum{}' \bar{b}_j^* \right| / \sum_1^N B_k^* \leqq \max_T \left| \sum{}' b_j^{*\prime} \right| / \sum_1^N B_k^* = \max_\phi F_N(\phi),$$

and the conclusion is seen to follow from Lemma 9.3 and Theorem 8. This establishes (vi).

THEOREM 11. *If* $\Phi_N(\phi) \equiv \sum_1^N |\cos(\phi - k)|/N$, $(0 \leqq \phi \leqq 2\pi)$, *then* $\lim_N \Phi_N(\phi) = 2/\pi$ *uniformly in* ϕ.

PROOF. As in the proof of Lemma 9.2, it can be shown that $\lim_N \cos_\phi \Phi_N(\phi) = 0$. Also,

$$\int_0^{2\pi} \Phi_N(\phi) d\phi = 4.$$

The conclusion follows.

REMARK. The sequence $\{b_k^*\}$ could equally well have been taken thus: $b_k^* = e^{i\delta k}$, $(k = 1, 2, \cdots)$, where δ is any number incommensurable with π.

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