ON THE ORDER OF THE PARTIAL SUMS OF A FOURIER SERIES*

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We propose to show here that the known estimate $S_n = o(n)$ cannot be improved. To do this it is sufficient to show that there exists a sequence of functions $f_n(x)$ for which there is a constant A such that,

$$(1)\qquad S_n(f_n, 0) > An,$$

and

$$(2)\qquad S_\nu(f_n, 0) \to 0 \quad \text{as} \quad \nu \to \infty,$$

$$(3)\qquad \int_{-\pi}^{\pi} | f_n(x)dx \,| < 1.$$

For, if (1), (2), and (3) are satisfied, then for every sequence of positive numbers d_n, with $d_n \to 0$ as $n \to \infty$, lim sup $nd_n = \infty$, we can choose a sequence of integers n_i such that

$$(4)\qquad \left| \sum_{j=1}^{i-1} d_{n_j} S_{n_i}(f_{n_j}, 0) \right| < \frac{A}{3} n_i d_{n_i}$$

and

$$(5)\qquad d_{n_{i+1}} < \frac{A}{3\pi} d_{n_i}.$$

We notice that

$$(6)\quad S_{n_j}(f_{n_i}, 0) = \frac{\pi}{2} \int_{-\pi}^{\pi} f_{n_i}(x) D_{n_j}(x) dx < \pi n_j \int_{-\pi}^{\pi} | f_{n_i}(x) | \, dx < \pi n_j,$$

and this implies that the constant A in (1) is less than π. Then, if $f(x)$ is defined by

$$f(x) = \sum_{i=1}^{\infty} d_{n_i} f_{n_i}(x),$$

$f(x) \subset L$, since from (3) and (5)

$$\int_{-\pi}^{\pi} | f(x) | \, dx \leqq \sum_{i=1}^{\infty} d_{n_i} \int_{-\pi}^{\pi} | f_{n_i}(x) | \, dx < \frac{d_1 A}{\pi} \sum_{i=1}^{\infty} 3^{-i} = \frac{d_1 A}{2\pi}.$$

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We have also

$$
\begin{aligned}
| S_{n_i}(f, 0) | &= \left| \sum_{j=1}^{\infty} d_{n_j} S_{n_i}(f_{n_j}, 0) \right| \\
&= \left| \sum_{j=1}^{i-1} d_{n_j} S_{n_i}(f_{n_j}, 0) + d_{n_i} S_{n_j}(f_{n_i}, 0) + \sum_{j=i+1}^{\infty} d_{n_j} S_{n_i}(f_{n_i}, 0) \right| \\
&> A n_i d_{n_i} - \tfrac{1}{3} A n_i d_{n_i} - \pi n_i \sum_{j=i+1}^{\infty} d_{n_j} \\
&> \tfrac{1}{3} A n_i d_{n_i}.
\end{aligned}
$$

We shall now prove the existence of a sequence $f_n(x)$ with the properties (1), (2), and (3). We define

$$
f_n(x) = \begin{cases} n/2, & \pi/4(n + \frac{1}{2}) \leqq | x | \leqq \pi/2(n + \frac{1}{2}), \\ 0, & \text{elsewhere}. \end{cases}
$$

Then

$$
(3) \qquad \int_{-\pi}^{\pi} | f_n(x) | \, dx = \frac{n\pi}{8(n + \frac{1}{2})} < 1,
$$

and, since $f_n(x)$ vanishes in the neighborhood of the origin,

$$
(2) \qquad S_\nu(f_n, 0) \to 0 \quad \text{as} \quad \nu \to \infty .
$$

Finally,

$$
\begin{aligned}
S_n(f_n, 0) &= \pi \int_0^{\pi} f_n(x) \frac{\sin (n + \frac{1}{2})x}{\sin x/2} dx \\
(1) \qquad &= \frac{n\pi}{2} \int_{\pi(n+1/2)/4}^{\pi/2(n+1/2)} \frac{\sin (n + \frac{1}{2})x}{\sin x/2} dx \\
&> \frac{\pi n}{2} \cdot \frac{\pi}{4(n + \frac{1}{2})} \cdot \frac{2(n + \frac{1}{2})}{\pi} > \pi n/4,
\end{aligned}
$$

and our theorem is proved.

We shall merely mention that similar results could be easily obtained for the Cesàro sums of a Fourier series. For the sequence of functions we have constructed satisfies (1), (2), and (3) when $S_n(f, x)$ is replaced by $S_n^\alpha(f, x)$ where $S_n^\alpha(f, x)$ is the C_α transform of the Fourier series of $f(x)$.

We shall now point out a consequence of this result.* We know, of

* The author is indebted to Professor Szász for pointing this out.

course, that the Fourier series of a function of bounded variation converges everywhere; we are now able to state that nothing can be said about the order of convergence even if we restrict ourselves to absolutely continuous functions. For let us suppose that there is some sequence c_n, $(c_n \to 0)$, such that for all absolutely continuous $F(x)$

$$S_n(F, x) - F(x) = O(C_n). \tag{7}$$

Let $f(x)$ be the function defined above, and define

$$f_a(x) = f(a + x) - f(a - x).$$

Then

$$f_a(x) \sim \sum_{n=1}^{\infty} b_n \sin nx$$

and

$$S_{n_i}(f_a, a) > \frac{A}{3} n_i d_{n_i} - o(1) \neq o(n_i d_{n_i}). \tag{8}$$

If $F(x)$ is defined by

$$F(x) = \int_0^x f(x) dx,$$

we have

$$F(x) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos nx,$$

where $A_n = b_n/n$, $n \geqq 1$. Then, if (7) is true,

$$\begin{aligned} S_n(f_a, x) &= \sum_{\nu=1}^{n} \nu A_\nu \sin \nu x \\ &= [- S_0(F, x) + F(x)] \sin \dot{x} + [S_n(F, x) - F(x)] n \sin nx \\ &\quad + \sum_{\nu=1}^{n-1} [S_\nu(F, x) - F(x)][\nu \sin \nu x - (\nu + 1) \sin (\nu + 1) x] \\ &= O(nC_n). \end{aligned}$$

But by (8) the d_n can be chosen so that this is impossible, and therefore (7) must be false.

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