and as above this is sufficient that postulates I–V be satisfied.

With this definition of $A:B$, $A \supset B$ becomes the usual inclusion relation of the algebra of classes [5].

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A NOTE ON THE MAXIMUM PRINCIPLE FOR ELLIPTIC DIFFERENTIAL EQUATIONS

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Let $u(x_1, \cdots, x_n)$ denote a twice continuously differentiable function of $x_1, \cdots, x_n$ in some region R. We write $\partial u/\partial x_i = u_i$, $\partial^2 u/\partial x_i \partial x_k = u_{ik}$, and occasionally (x) for $(x_1, \cdots, x_n)$. A point $(c) = (c_1, \cdots, c_n)$ of R may be called a *proper* maximum of u, if

$$u_i(c) = 0 \qquad \text{for} \qquad i = 1, \cdots, n,$$

$$\sum_{i,k} u_{ik}(c)\xi_i\xi_k < 0 \qquad \text{for all} \qquad (\xi_1, \cdots, \xi_n) \neq (0, \cdots, 0).$$

A partial differential equation

$$\sum_{i,k} a_{ik}(x)u_{ik}(x) + \sum_i b_i(x)u_i(x) = 0 \tag{1}$$

(where the a_{ik} and b_i are defined in R) is called *elliptic* if for every (x) of R

$$\sum_{i,k} a_{ik}(x)\xi_i\xi_k \geqq 0$$

for all $(\xi_1, \cdots, \xi_n)$ and <0 for some $(\xi_1, \cdots, \xi_n)$.

It is well known, that a solution u of (1) can not have a proper maximum.* For if u had a proper maximum at $(c_1, \cdots, c_n)$, then $\sum_{i,k} a_{ik}(c)u_{ik}(c)=0$. If A and U denote respectively the matrices $(a_{ik}(c))$ and $(u_{ik}(c))$, this may be written: Trace $(A \cdot U)=0$. By a suitable orthogonal transformation A may be transformed into a diagonal matrix $A' = (a_i' \delta_{ik})$, U going over into $U' = (u'_{ik})$ by the same transformation. As the trace of $A \cdot U$ is preserved, we have $\sum_i a_i' u'_{ii} = 0$; on the other hand, as A' still belongs to a semi-definite quadratic form and U' to a negative definite one, we have $a_i' \geqq 0$ for $i=1, \cdots, n$, but <0 for some i, and $u'_{ii}<0$ for all i. This leads to a contradiction.

A second important property of the solutions u of (1) is that they form a module, that is, that every linear combination with constant coefficients is again a solution.

We shall prove that these two properties are also sufficient to characterize a family of functions as solutions of an elliptic equation (1).

THEOREM. *Let F be any family of twice continuously differentiable functions $u(x_1, \cdots, x_n)$ defined in R, such that*

(a) *the functions of F form a module,*

(b) *no function of F has a proper maximum.*

Then there is an elliptic differential equation (1) *satisfied by all functions of F.*

PROOF. Let $(c)=(c_1, \cdots, c_n)$ be a point of R. Let ϕ be the submodule of functions u of F for which $u_i(c)=0$ for $i=1, \cdots, n$. Let $Q(\xi_1, \cdots, \xi_n)$ denote the quadratic form $\sum_{i,k} u_{ik}(c)\xi_i\xi_k$ for any u in ϕ; Q is certainly not negative definite. These quadratic forms form again a module M. Let $Q_1, Q_2, \cdots, Q_m$ form a basis of this module $(m \leqq n(n+1)/2)$, such that for every Q of M

$$Q(\xi_1, \cdots, \xi_n) = \sum_{i=1}^{m} \lambda_i Q_i(\xi_1, \cdots, \xi_n)$$

with certain constants λ_i. We know that for every $(\lambda_1, \cdots, \lambda_m)$ there are $(\xi_1, \cdots, \xi_n) \neq (0, \cdots, 0)$ such that

$$\sum_{i=1}^{m} \lambda_i Q_i(\xi_1, \cdots, \xi_n) \geqq 0. \tag{2}$$

From this we can easily conclude that the Q_i satisfy a linear relation

* Cf. Encyklopädie der Mathematischen Wissenschaften, vol. 2, 1.1, p. 522, or Picard, *Traité d'Analyse*, 3d ed., vol. 2, p. 29 for the case $n=2$. The subsequent proof follows Courant-Hilbert, *Methoden der Mathematischen Physik*, vol. 2.

with positive coefficients.* For let Σ denote the set of points with coordinates $(Q_1(\xi), Q_2(\xi), \cdots, Q_m(\xi))$ in an m-dimensional space, where $(\xi)=(\xi_1, \cdots, \xi_n)$ varies over all points of the unit sphere $\xi_1^2+\xi_2^2+\cdots+\xi_n^2=1$. The set Σ is closed and finite. The relation (2) may be interpreted as stating that every half-space bounded by a plane through the origin contains points of Σ. Thus the origin is contained in the convex extension of Σ. Then there exists a finite set of points $P_1, \cdots, P_r$ of Σ and positive numbers $\mu_1, \cdots, \mu_r$, such that the origin is the center of mass of the masses μ_i placed at the vertices P_i.† Let $(\xi_1^i, \cdots, \xi_n^i)$ be the point $(\xi_1, \cdots, \xi_n)$ corresponding to P_i. Then

$$\sum_{j=1}^{r} \mu_j Q_k(\xi_1^j, \cdots, \xi_n^j) = 0$$

for $k=1, \cdots, m$, and consequently

$$\sum_{j=1}^{r} \mu_j Q(\xi_1^j, \cdots, \xi_n^j) = 0$$

for every Q in M. Thus

$$\sum_{j=1}^{r} \sum_{i,k} \mu_j u_{ik}(c_1, \cdots, c_n)\xi_i^j \xi_k^j = 0$$

for u in ϕ. Putting $\sum_j \mu_j \xi_i^j \xi_k^j = a_{ik}(c)$, we have

$$\sum_{i,k} a_{ik}(c) u_{ik}(c) = 0$$

for u in ϕ. Besides

$$\sum_{i,k} a_{ik}(c)\xi_i\xi_k = \sum_{i,j} \mu_j(\xi_i^j \xi_i)^2 \geqq 0$$

and >0 for some $(\xi_1, \cdots, \xi_n)$.

Now let $u(x_1, \cdots, x_n)$ denote an arbitrary function of F. The vectors $(y_1, \cdots, y_n)=(u_1(c), u_2(c), \cdots, u_n(c))$ again form a module N, if u varies over F for fixed (c). Let $(y_1^1, y_2^1, \cdots, y_n^1)$, $(y_1^2, \cdots, y_n^2)$, $\cdots$, $(y_1^s, \cdots, y_n^s)$ form a basis of N, $(s \leqq n)$. Without restriction of generality we may assume that this basis forms an orthogonal system:

$$\sum_{l=1}^{n} y_l^j y_l^k = \delta_{ik}, \qquad i, k = 1, \cdots, s.$$

* Cf. L. L. Dines, this Bulletin, vol. 42, p. 357, and the paper of R. W. Stokes Transactions of this Society, vol. 33, p. 782 et seq.

† Bonnesen-Fenchel, *Theorie der Konvexen Körper*, p. 9.

Let $u^1(x, c), u^2(x, c), \cdots, u^s(x, c)$ be the functions of F corresponding to the vectors

$$u_i^k(c, c) = y_i^k.$$

Then for every u in F

$$u_i(c) = \sum_{j=1}^{s} \lambda_j y_i^j$$

with

$$\lambda_j = \sum_{k=1}^{n} u_k(c) y_k^j.$$

Thus

$$u_i(c) = \sum_{j=1}^{s} \sum_{k=1}^{n} u_k(c) u_k^j(c, c) u_i^j(c, c).$$

Consider the function

$$\bar{u}(x) = u(x) - \sum_{j=1}^{s} \sum_{k=1}^{n} u_k(c) u_k^j(c, c) u^j(x, c).$$

Then $\bar{u}$ is in F. We have

$$\bar{u}_i(c) = u_i(c) - \sum_{j=1}^{s} \sum_{k=1}^{n} u_k(c) u_k^j(c, c) u_i^j(c, c) = 0.$$

Hence $\bar{u}$ is in ϕ. Consequently

$$0 = \sum_{i,k} a_{ih}(c) \bar{u}_{ih}(c) = \sum_{i,h} a_{ih}(c) u_{ih}(c) - \sum_{i,j,k,h} a_{ih}(c) u_k(c) u_k^j(c, c) u_{ih}^j(c, c).$$

Thus $u(x)$ satisfies the elliptic equation

$$0 = \sum_{i,h} a_{ih}(x) u_{ih}(x) - \sum_{i,j,k,h} a_{ih}(x) u_k^j(x, x) u_{ih}^j(x, x) u_k(x).$$

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