THE RESOLVENT OF A CLOSED TRANSFORMATION

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1. **Introduction.** In reading over Chapter IV of Stone's book, *Linear Transformations in Hilbert Space*, I was impressed by the fact that a number of the results obtained are valid for any complex Banach space. This generality does not always appear at once evident, and it may be worth while to explain briefly.

The most interesting and important fact which underlies the material is that the resolvent of a closed distributive* transformation depends *analytically* on a parameter λ. This dependence is made precise in Stone's work with the aid of the inner product of Hilbert space; but this is not necessary, for it is known that the fundamental portions of the classical theory of analytic functions remain valid in complex Banach spaces.† In particular, Liouville's theorem admits a valid generalization. Thus we are able to prove that the spectrum of a (continuous) linear transformation whose domain is the whole space E is not empty. We shall now turn to the details.

2. **Preliminaries.** We use E to denote a complex Banach space; T will denote a distributive (additive, homogeneous) transformation, with domain and range both in E. We then write $T_\lambda = T - \lambda I$, and T_λ^{-1} will denote the inverse of T_λ when it exists. Here λ is a complex number, and I the identity transformation. We recall that a transformation admits an inverse if and only if it sets up a one to one correspondence between its domain and its range. When T is distributive, the necessary and sufficient condition that T_λ^{-1} exist is that $T_\lambda f = 0$ imply $f = 0$. The set of values of λ for which T_λ^{-1} is linear, with domain everywhere dense in E, is called the resolvent set of T. All other values of λ constitute the spectrum of T.

3. **Discussion of the resolvent.** We prove the following theorem:

THEOREM 1. *If $T_{\lambda_0}^{-1}$ exists and is linear, then T_λ^{-1} exists and is linear for each λ such that $|\lambda - \lambda_0| < 1/M_{\lambda_0}$, where M_{λ_0} is the modulus of $T_{\lambda_0}^{-1}$.*

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* We use *distributive* where Stone uses *linear*, preferring to reserve the latter term for continuous distributive transformations. For the definition of a closed transformation see Stone, loc. cit., p. 38.

† For the independent variable a complex number this was pointed out by Wiener, Fundamenta Mathematicae, vol. 4 (1923). For the general case see A. E. Taylor, Comptes Rendus, vol. 203 (1936), pp. 1228–1230, and a forthcoming paper in the Annali della Reale Scuola Normale di Pisa; also L. M. Graves, this Bulletin, vol. 41 (1935), pp. 651–653.

PROOF. Let f be the domain of T. Then $T_\lambda f = T_{\lambda_0} f - (\lambda - \lambda_0) f$. But since $T_{\lambda_0}^{-1}$ is linear, $\|f\| \leqq M_{\lambda_0} \|T_{\lambda_0} f\|$, and hence if $|\lambda - \lambda_0| < 1/M_{\lambda_0}$, $T_\lambda f = 0$ implies $f = 0$, so that T_λ^{-1} exists. Also, since

$$\|Tf - \lambda f\| \geqq \|T_{\lambda_0} f\| - |\lambda - \lambda_0| \, \|f\| \geqq \left(\frac{1}{M_{\lambda_0}} - |\lambda - \lambda_0| \right) \|f\|,$$

we conclude that T_λ^{-1} is bounded, and hence linear. Its modulus does not exceed $M_{\lambda_0}/(1 - M_{\lambda_0}|\lambda - \lambda_0|)$.

THEOREM 2. *If T is closed, and if its resolvent set is not empty, the domain of T_λ^{-1} is the whole space E when λ is in the resolvent set.*

The proof of this offers no difficulties, and we omit it. The family of linear transformations T_λ^{-1} is called the resolvent. We denote it by R_λ.

The next theorem is a direct carry over from Stone (Theorem 4.10, p. 137). The proof given by Stone is valid in the present case, and the reader is referred to it.

THEOREM 3. *If the resolvent R_λ of a closed distributive transformation T exists, then $R_\lambda - R_\mu \equiv (\lambda - \mu) R_\lambda R_\mu$ throughout E for each pair of values λ, μ in the resolvent set, and $R_\lambda f = 0$ implies $f = 0$.*

Conversely, if X_λ is a family of linear transformations with domain E, defined for each λ in a set Σ, such that

$$X_\lambda - X_\mu \equiv (\lambda - \mu) X_\lambda X_\mu \tag{1}$$

for each λ, μ in Σ; and if $X_\lambda f = 0$ implies $f = 0$ for at least one λ in Σ, then there exists a unique closed, distributive transformation T whose resolvent exists and coincides with X_λ for every λ in Σ.

The functional equation (1) is striking. It suggests at once a "law of the mean" for $X_\lambda f$, and in a sense is just that. We shall consider some further conclusions which may be drawn from (1) under suitable hypotheses.

THEOREM 4. *Let X_λ be a linear transformation with domain E for each λ in an open set Σ of the complex plane; further, let $X_\lambda f$ be continuous in Σ for each f in E, and suppose that the functional equation (1) is satisfied when λ, μ are in Σ. Then $X_\lambda f$ is analytic* in Σ for each f in E, and in the neighborhood of each point of Σ admits an expansion in the form*

* In the sense indicated by Wiener. See the references in an earlier footnote.

(2) $$X_\lambda f = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n X_{\lambda_0}^{n+1} f,$$

where λ_0 *is in* Σ *and* $|\lambda - \lambda_0|$ *is sufficiently small.*

PROOF. We first prove that $X_\lambda f$ admits a derivative with respect to λ at each point of Σ. The form of (1) suggests that this derivative is $X_\lambda^2 f$; this surmise is verified by the inequality

$$\left\| \frac{X_\lambda f - X_{\lambda_0} f}{\lambda - \lambda_0} - X_{\lambda_0}^2 f \right\| = \| X_{\lambda_0}(X_\lambda f - X_{\lambda_0} f)\| \leqq M_{\lambda_0} \| X_\lambda f - X_{\lambda_0} f \|,$$

where M_{λ_0} is the modulus of X_{λ_0}, since $X_\lambda f$ is continuous with respect to λ (it is easily seen that $X_\lambda X_\mu \equiv X_\mu X_\lambda$). It follows from the general theory of abstract-valued analytic functions that $X_\lambda f$ is analytic in Σ. In particular, it admits derivatives of all orders in Σ, and is expansible in a Taylor series. In order to show that this expansion about a point $\lambda = \lambda_0$ has the form (2), we shall prove that the derivatives of $X_\lambda f$ are given by the formula

(3) $$[X_\lambda f]^{(n)} \equiv \frac{d^n}{d\lambda^n} [X_\lambda f] = n! X_\lambda^{n+1} f.$$

This has already been established for $n = 1$. We proceed by induction, assuming the truth of (3) with n replaced by $n - 1$. Then

$$\left\| \frac{[X_\lambda f]^{(n-1)} - [X_{\lambda_0} f]^{(n-1)}}{\lambda - \lambda_0} - n! X_{\lambda_0}^{n+1} f \right\|$$

$$= \left\| (n-1)! \frac{X_\lambda^n f - X_{\lambda_0}^n f}{\lambda - \lambda_0} - n! X_{\lambda_0}^{n+1} f \right\|$$

$$= \left\| (n-1)! \frac{X_\lambda - X_{\lambda_0}}{\lambda - \lambda_0} (X_\lambda^{n-1} + X_\lambda^{n-2} X_{\lambda_0} + \cdots + X_{\lambda_0}^{n-1}) f - n! X_{\lambda_0}^{n+1} f \right\|$$

$$= \| (n-1)! X_{\lambda_0} X_\lambda (X_\lambda^{n-1} + X_\lambda^{n-2} X_{\lambda_0} + \cdots + X_{\lambda_0}^{n-1}) f - n! X_{\lambda_0}^{n+1} f \|$$

$$\leqq (n-1)! M_{\lambda_0} \| (X_\lambda^n + X_\lambda^{n-1} X_{\lambda_0} + \cdots + X_\lambda X_{\lambda_0}^{n-1}) f - n X_{\lambda_0}^n f \|.$$

But the expression inside the last norm sign tends to zero as $\lambda \to \lambda_0$, by virtue of the continuity of X_λ, and so the induction is completed. The continuity of the iterates of X_λ, and therefore of the expression inside the norm, is deduced from the fact that $X_\lambda f$, being linear in f, is continuous in λ and f *together*.*

* This is a theorem of Kerner, Studia Mathematica, vol. 3 (1931), p. 159. Formula (3) could also be established by use of some theorems about Fréchet differentials. For the above direct proof I am indebted to the referee.

From (3) and the fact that Σ is an open set we can infer the validity of (2) for some region $|\lambda-\lambda_0|<\rho$. This completes the proof of Theorem 4.

However, it is apparent that (2) converges and defines a linear transformation with domain E whenever $|\lambda-\lambda_0|<1/M_{\lambda_0}$. If this range of values was not originally included in Σ, the domain of definition of X_λ may be extended, and it is easy to see that (1) will continue to be satisfied.

The foregoing considerations suggest at once the nature of the resolvent of T.

THEOREM 5. *Let T be a closed, distributive transformation whose resolvent exists. Then the resolvent set Σ is an open set, and the resolvent $R_\lambda f$ is an analytic function of λ in Σ. If M_λ is the modulus of R_λ, Σ is such that with λ_0 it contains all points λ such that $|\lambda-\lambda_0|<1/M_{\lambda_0}$. R_λ is then given by the expansion*

$$R_\lambda f = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n R_{\lambda_0}^{n+1} f \tag{4}$$

and its modulus satisfies the inequality

$$M_\lambda \leqq \frac{M_{\lambda_0}}{1 - |\lambda - \lambda_0| M_{\lambda_0}}. \tag{5}$$

PROOF. Since Σ is non-empty, it contains a point λ_0, and R_{λ_0} is a linear transformation with domain E (Theorem 2). Let us define a family of transformations $X_\lambda f$ by the series on the right in equation (4). It is clear that $X_\lambda f$ will be linear, with domain E, when $|\lambda-\lambda_0|<1/M_{\lambda_0}$. Next we identify X_λ with T_λ^{-1} when $|\lambda-\lambda_0|<1/M_{\lambda_0}$. The calculations are identical with those in Stone, pages 140–141, and will be omitted. By definition, then, the resolvent R_λ of T exists and coincides with X_λ when $|\lambda-\lambda_0|<1/M_{\lambda_0}$. The resolvent set is accordingly open. Equation (4) now gives $R_\lambda f$ as a power series with abstract coefficients; it is therefore analytic as a function of λ. This result may be obtained without further resort to the theory of abstract power series by appealing to Theorem 4, since $R_\lambda f$ is now evidently continuous as a function of λ, and satisfies the functional equation (1), by Theorem 3. The inequality (5) has already been noted in Theorem 1. Theorem 5 is thus proved.

4. **The resolvent of a linear transformation.** We state now another theorem:

THEOREM 6. *If T is linear, with domain E and modulus C, the resolvent set contains all λ such that $|\lambda| > C$, and for these values*

$$(6) \qquad R_\lambda f = -\sum_{n=1}^{\infty} \frac{1}{\lambda^n} T^{n-1} f.$$

Thus all the singularities of $R_\lambda f$ lie in the finite part of the plane. In particular $R_\lambda f$ has at least one singularity, that is, the spectrum of T is not empty.

The proof of (6) is well known. The last assertion of the theorem follows from Liouville's theorem (for abstract analytic functions), since from (6) we have

$$\|R_\lambda f\| \leqq \frac{\|f\|}{|\lambda| - C}, \qquad |\lambda| > C.$$

If the spectrum of T were empty, $R_\lambda f$ (f fixed) would be analytic over the entire plane and finite at infinity. Then $R_\lambda f$ would be a constant, with value 0, for all f, since $\lim_{|\lambda|\to\infty} \|R_\lambda f\| = 0$. This is impossible. (We exclude, of course, the trivial case where E consists of the zero element alone.)

It would be interesting to know other relationships between the nature of T and the singularities of its resolvent.

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