REMES ON APPROXIMATION

On Methods for Obtaining the Best Approximation of Functions, in the Sense of Tchebycheff. By E. Remes (in the Ukrainian language). Mathematical Institute of the Ukrainian Academy of Sciences, Kieff, 1935. i+159 pp.

This is one of a series of mathematical monographs being published in Soviet Russia. Its main object is the study of polynomial—more particularly, "best" polynomial, in the sense of Tchebycheff—approximation of various classes of functions. Considerable space is devoted to functions of several variables—a fertile field awaiting further cultivation. Following the general utilitarian tendencies of science in Soviet Russia, the theoretical discussion is in places rather condensed. To compensate, methods are developed for the actual computation of the approximating polynomials, which are applied to concrete examples, the error involved being also given. Thus the methods and results are made usable in engineering practice.

The book is divided into two parts. Part I deals with problems in the theory of approximation which admit solutions in a closed form. It consists of the following five sections: §1. Approximate representation of a function $f(x, y, \cdots, v)$ defined in a certain region, by means of linear functions $\phi_1 = \alpha_0 + \alpha_1 x + \cdots$ or functions of the type $\phi_2 = \alpha_0 + \alpha_1 x + \cdots + \alpha_{1,2} xy + \cdots + \alpha_{1,2 \ldots} xy \cdots v$ (choice of the above parameters, relative and absolute error). Application to harmonic, sub- and super-harmonic functions ($\Delta^2 f \equiv \partial^2 f/\partial x^2 + \cdots + \partial^2 f/\partial v^2 = 0$, $\geqq 0$, $\leqq 0$ in a certain region). Special consideration is given to Tchebycheff's Map-Problem which essentially reduces to the following one: Find a harmonic function $\Phi(x, y)$ with the least possible deviation from $f(x, y) = 2/(e^x + e^{-x})$ in a certain region bounded by a continuous contour C. Show that $\Phi(x, y)$ differs by a constant from that harmonic function which on C takes on values given by the above $f(x, y)$. §§2, 3. Approximate representation of functions w defined by equations of the type $w^\rho = x^\rho + y^\rho + \cdots + v^\rho$, that is,

$$(x^\rho + y^\rho + \cdots + v^\rho)^{1/\rho} \cong \alpha x + \beta y + \cdots + \kappa v.$$

The case $\rho = 2$ deserves—and receives—special consideration. In fact, this is the celebrated Poncelet problem: Find the best possible linear approximation for $(x^2 \pm y^2)^{1/2}$, $(x^2+y^2+z^2)^{1/2}$; more generally, for $(x_1^2 + \cdots + x_n^2)^{1/2}$. It was this problem which in the masterly hands of Tchebycheff became the cornerstone of the theory of best approximation. The book deals with this problem in §§2 and 3, analytically and geometrically, for $n = 1, 2, \cdots, 10$. A few results follow, as an illustration.

$$(x_1^2 + x_2^2)^{1/2} \cong .960x_1 + .398x_2, \qquad \text{within } 4\%,$$

$$(x_1^2 + x_2^2 + x_3^2)^{1/2} \cong .940x_1 + .328x_2 + .299x_3, \qquad \text{within } 6\%,$$

$$x_1 \geqq x_2 \geqq \cdots \geqq x_{10} \geqq 0.$$

$$(1 + y^2)^{1/2} \cong .955 + .414y, \quad \text{absolute error} < .045, \qquad 1 \geqq y \geqq z \geqq 0.$$

$$(1 + y^2 + z^2)^{1/2} \cong .926 + .414y + .318z, \quad \text{absolute error} < .074.$$

§4. Linear approximations for algebraic rational fractions and irrational functions. Application is then made to differential equations (illustration: $dy/dx = x^{1/2} + y^{1/2}$). The relation of the above considerations is shown—rather sketchily—to Bernoulli's method for obtaining that root of an algebraic equation whose modulus is largest. The formulas given in §2 are further applied to Graeffe's method for solving algebraic equations. This is developed with more details and computations. §5. The best approximation to a convex function by means of the ordinates of a polygonal line, with a preassigned number of vertices. Best linear approximation for x^p, xy, xyz, $\cdots$. Approximation of transcendental functions and solution of transcendental equations. Approximation of functions by means of the above functions ϕ_2. Thus,

$$e^x \cong .8459x^2 + .9549x + 1.0007, \qquad \text{error} < .088, 0 \leqq x \leqq 1,$$

$$e^{(x+y)/2} \cong .946 + .649(x + y) + .421xy, \quad \text{error} < .054, 0 \leqq x, y \leqq 1,$$

and, with the last term "linearized,"

$$e^{(x+y)/2} \cong .908 + .638x + 1.050y. \tag{1}$$

Part II deals with general methods for obtaining the "best" polynomial approximation for functions in one or several variables. It consists of three sections: §6. Characterization and general properties of the polynomial of best approximation to a function $f(x)$ defined on a given set $E<(a, b)$; existence, bounds for the best approximation, $\cdots$. The discussion follows in the main that of the well known book of de la Vallée-Poussin on approximation. §7. Application of the method of successive approximations in order to obtain the polynomial in question. The author develops two algorithms. The second seems to us to be the more interesting one. It is based upon first finding the polynomial of best approximation, of degree $\leqq n$, to $f(x)$—assumed to be given on (a, b)—corresponding to $n+2$ points arbitrarily chosen on (a, b), and then introducing proper corrections. The above methods are applied to $f(x) = |x|$, $|x| + \kappa \operatorname{sgn} x$, $\kappa > 0$. Thus, the polynomial $.0675 + 1.9313x^2 - 1.0665x^4$ represents $|x|$ on $(-1, 1)$, within 0.0677; the difference between this approximation and the best possible one (the degree of the polynomial being $\leqq 4$) is $< 10^{-4}$. §8. This section deals with systems of linear equations and with the best linear approximation of functions in several variables. The author emphasizes the importance of this section as was stated above. The method of successive approximations is sketched. Use is made of the results obtained in Part I. Thus, in order to obtain the best linear approximation to $e^{(x+y)/2}$ in the region $1 \geqq x \geqq y \geqq 0$ we start with the above approximation (1), and seek to determine the necessary corrections. In this way we find that the best approximation ρ has the value $\rho = .106$, (error of order 10^{-4}).

The book closes with a treatment of the best approximation of empirical functions (given by a set of data) by means of linear aggregates of functions of preassigned type.

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