$$\frac{\partial P_i}{\partial \sigma} = A_i. \tag{14}$$

This system, together with the initial conditions, is satisfied by $P_i=0$, $(i=1, \cdots, k)$. Hence, on account of the uniqueness of the solution of (14) with given initial values, we conclude that $P_i \equiv 0$, and the proof is complete.

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# ON THE EXISTENCE OF LINEAR FUNCTIONALS DEFINED OVER LINEAR SPACES*

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1. *Introduction.* A function $q(x)$ with domain in a linear space $E$ and range in the set $R$ of real numbers is called a *functional*, and $q(x)$ is called *linear*, if

$$q(ax + by) = aq(x) + bq(y), \qquad x,y \varepsilon E;\ a,b \varepsilon R. \tag{1}$$

We call a functional $r(x)$ an *r-function* (over $E$) if there exists a linear functional $f(x)$ with

$$f(x) \leqq r(x), \qquad x \varepsilon E. \tag{2}$$

Using a notation of Banach† we call a functional $p(x)$ a *p-function* if

$$p(tx) = tp(x), \qquad t \geqq 0,\ x \varepsilon E, \tag{3}$$

$$p(x + y) \leqq p(x) + p(y), \qquad x,y \varepsilon E. \tag{4}$$

A fundamental theorem of Banach (loc. cit., p. 29) can be stated as follows:

THEOREM (Banach). *Each p-function is an r-function.*

In some problems‡ involving existence of linear functionals $f_1(x)$ having prescribed properties, there appears a functional $q(x)$ with the following significance: There exists a linear functional $f_1$ having the requisite properties if and only if there exists

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* Presented to the Society, September 8, 1937.

† S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 28.

‡ The author intends to discuss these problems at some future time.

a linear functional $f$ with $f(x) \leqq q(x)$, that is, if and only if $q(x)$ is an $r$-function. If $q(x)$ can be shown to be a $p$-function, the problem is solved by Banach's theorem; if $q(x)$ is not a $p$-function or one is unable to decide whether $q(x)$ is a $p$-function, Banach's theorem cannot be applied. These considerations, and the fact that it is easy to give examples of $r$-functions which are not $q$-functions, lead one to desire an analytic characterization of $r$-functions. In §2 we give such a characterization, and in §3 we give some closely related theorems.

2. *Characterization of r-functions.* We prove now the theorem:

THEOREM 1. *In order that a functional $r(x)$ defined over $E$ may be an r-function, it is necessary and sufficient that*

$$\underset{n,t_k>0;\Sigma x_k=0}{\text{g.l.b.}} \sum_{k=1}^{n} \frac{r(t_k x_k)}{t_k} \geqq 0. \tag{5}$$

In (5), $\sum x_k$ stands for the sum $x_1 + \cdots + x_n$ of elements $x_k \varepsilon E$. To prove necessity, let $r(x)$ be an $r$-function and let $f(x)$ be a linear functional with $f(x) \leqq r(x)$ for all $x \varepsilon E$. Then if $n, t_1, t_2, \cdots, t_n > 0$ and $\sum x_k = 0$, we have

$$f(x_k) = f(t_k x_k)/t_k \leqq r(t_k x_k)/t_k, \tag{6}$$

so that

$$0 = f(0) = f(\sum x_k) = \sum f(x_k) \leqq \sum r(t_k x_k)/t_k, \tag{7}$$

and (5) follows.

To prove sufficiency, let (5) hold and define the functional $p^{(r)}(x)$ by the formula

$$p^{(r)}(x) = \underset{n,t_k>0;\Sigma x_k=x}{\text{g.l.b.}} \sum_{k=1}^{n} \frac{r(t_k x_k)}{t_k}, \qquad x \varepsilon E. \tag{8}$$

To show that $p^{(r)}(x)$ exists (is finite) for each $x \varepsilon E$, we observe that if $n, t_1, \cdots, t_n > 0$ and $\sum x_k = x$, then $x_1 + \cdots + x_n + (-x) = 0$ and it follows from (5) that

$$\sum_{k=1}^{n} \frac{r(t_k x_k)}{t_k} + \frac{r(-x)}{1} \geqq 0,$$

and hence

$$- r(- x) \leqq \sum_{k=1}^{n} \frac{r(t_k x_k)}{t_k},$$

which implies that $-r(-x) \leqq p^{(r)}(x)$. If in the sum in the right member of (8) we put $n=1$, $t_1=1$, $x_1=x$, we obtain $p^{(r)}(x) \leqq r(x)$. Therefore

$$(9) \qquad - r(- x) \leqq p^{(r)}(x) \leqq r(x), \qquad x \varepsilon E.$$

We prove next that $p^{(r)}(x)$ is a $p$-function. If $x \varepsilon E$ and $t>0$, then

$$\begin{aligned} p^{(r)}(tx) &= \operatorname*{g.l.b.}_{n,t_k>0;\Sigma x_k=tx} \sum_{k=1}^{n} \frac{r(t_k x_k)}{t_k} \\ &= t \operatorname*{g.l.b.}_{n,tt_k>0;\Sigma(x_k/t)=x} \sum_{k=1}^{n} \frac{r[(tt_k)(x_k/t)]}{tt_k} \\ &= t \operatorname*{g.l.b.}_{n,u_k>0;\Sigma y_k=x} \sum_{k=1}^{n} \frac{r(u_k y_k)}{u_k} = tp^{(r)}(x), \end{aligned}$$

so that $p^{(r)}(tx)=tp^{(r)}(x)$ for $t>0$. Substitution of $t=2$, $x=0$ in this formula gives $p^{(r)}(0)=0$. Therefore

$$(10) \qquad p^{(r)}(tx) = tp^{(r)}(x), \qquad t \geqq 0;\ x \varepsilon E.$$

To prove that

$$(11) \qquad p^{(r)}(x + y) \leqq p^{(r)}(x) + p^{(r)}(y), \qquad x,y \varepsilon E,$$

let $x,y\varepsilon E$ be fixed and let $\epsilon=0$. Choose $m$, $t_1, \cdots, t_m>0$ and $x_1, \cdots, x_m \varepsilon E$ such that $\sum x_j=x$ and

$$\sum_{j=1}^{m} r(t_j x_j)/t_j < p^{(r)}(x) + \epsilon;$$

and choose $n$, $u_1, \cdots, u_n>0$ and $y_1, \cdots, y_n \varepsilon E$ such that $\sum y_k=y$ and

$$\sum_{k=1}^{n} r(u_k y_k)/u_k < p^{(r)}(y) + \epsilon.$$

Since $m+n$, $t_j$, $u_k>0$ and $x_1+ \cdots +x_m+y_1+ \cdots +y_n=x+y$, it follows from the definition of $p^{(r)}(x+y)$ that

$$p^{(r)}(x + y) \leqq \sum_{j=1}^{m} \frac{r(t_j x_j)}{t_j} + \sum_{k=1}^{n} \frac{r(u_k y_k)}{u_k} < p^{(r)}(x) + p^{(r)}(y) + 2\epsilon.$$

The arbitrariness of $\epsilon > 0$ gives (11). Thus $p^{(r)}(x)$ is a $p$-function and it follows from Banach's theorem that there exists a linear functional $f(x)$ with $f(x) \leqq p^{(r)}(x)$. Using (9), we obtain $f(x) \leqq r(x)$; thus $r(x)$ is an $r$-function and Theorem 1 is proved.

3. *Significance of* $p^{(r)}(x)$. From Theorem 1 and its proof, we obtain the first part of our next theorem.

THEOREM 2. *If $r(x)$ is an $r$-function, then the functional $p^{(r)}(x)$ defined by*

$$p^{(r)}(x) = \underset{n, t_k > 0; \Sigma x_k = x}{\text{g.l.b.}} \sum_{k=1}^{n} \frac{r(t_k x_k)}{t_k}, \qquad x \varepsilon E, \tag{12}$$

*is a $p$-function with*

$$- r(- x) \leqq - p^{(r)}(x) \leqq p^{(r)}(x) \leqq r(x), \qquad x \varepsilon E; \tag{13}$$

*moreover if $p(x)$ is a $p$-function with $p(x) \leqq r(x)$ for all $x \varepsilon E$, then*

$$- p^{(r)}(- x) \leqq - p(- x) \leqq p(x) \leqq p^{(r)}(x), \qquad x \varepsilon E. \tag{14}$$

In establishing (13), we use (9) and the fact that, for any $p$-function, $0 = p(0) = p(x - x) \leqq p(x) + p(-x)$ and hence $-p(-x) \leqq p(x)$ for all $x \varepsilon E$. If $p(x) \leqq r(x)$; $n$, $t_1, \cdots, t_n > 0$; and $\sum x_k = x$; then

$$p(x) \leqq \sum_{k=1}^{n} p(x_k) = \sum_{k=1}^{n} p(t_k x_k)/t_k \leqq \sum_{k=1}^{n} r(t_k x_k)/t_k$$

and $p(x) \leqq p^{(r)}(x)$ follows. The remaining inequalities in (14) follow easily, and Theorem 2 is proved. The gist of Theorem 2 is that $p^{(r)}(x)$ is the "greatest" $p$-function $p(x)$ with $p(x) \leqq r(x)$. In particular, if $r(x)$ is a $p$-function, then $p^{(r)}(x) = r(x)$.

Since each linear functional $f(x)$ is a $p$-function, Theorem 2 implies the following theorem:

THEOREM 3. *If $r(x)$ is an $r$-function and $f(x)$ is a linear functional with $f(x) \leqq r(x)$, then*

$$- r(- x) \leqq - p^{(r)}(- x) \leqq f(x) \leqq p^{(r)}(x) \leqq r(x), \quad x \varepsilon E. \tag{15}$$

It thus appears that the class of linear functionals $f(x)$ for which $f(x) \leqq p^{(r)}(x)$ is identical with the class of linear functionals $f(x)$ for which $f(x) \leqq r(x)$.

4. *Conclusion.* The functionals $q(x)$ mentioned in the introduction often have the property $q(tx) = tq(x)$ for $t \geqq 0$, and $x \varepsilon E$. Hence it is of interest to note that if

$$r(tx) = tr(x), \qquad t \geqq 0, \qquad x \varepsilon E, \tag{16}$$

then the criterion (5) that $r(x)$ be an $r$-function reduces to

$$\underset{n>0;\Sigma x_k=0}{\text{g.l.b.}} \sum_{k=1}^{n} r(x_k) \geqq 0, \tag{17}$$

and that formula (12) for $p^{(r)}(x)$ reduces to

$$p^{(r)}(x) = \underset{n>0;\Sigma x_k=x}{\text{g.l.b.}} \sum_{k=1}^{n} r(x_k), \qquad x \varepsilon E. \tag{18}$$

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