CONCERNING NORMAL AND COMPLETELY NORMAL SPACES*

BY F. B. JONES

Urysohn has shown that any completely separable, normal topological space is metric. It is the principal object of this paper to establish a similar result for certain separable spaces.

THEOREM 1. *Every subset of power c† of a separable normal‡ Fréchet space-L (or -H) has a limit point.*

PROOF. Suppose, on the contrary, that S is a separable, normal Fréchet space-L (or -H) which contains a point set M of power c having no limit point. Let Z denote a countable subset of S such that every point of S either belongs to Z or is a limit point of Z. Since S is normal, there exists for each proper subset J of M a domain D_J which contains J but which neither contains a point of $M-J$ nor has a limit point in $M-J$. If J and K are two different proper subsets of M, then $Z\cdot D_J$ and $Z\cdot D_K$ are different subsets of Z. Hence, there are at least as many subsets of Z as there are proper subsets of M. However, since M is of power c and Z is only countable, there are *more* than c proper subsets of M but at most c subsets of Z. This is a contradiction.

The above argument with slight changes establishes the following three theorems.

THEOREM 2. *Every subset of power c of a separable, completely normal§ Fréchet space-L (or -H) contains a limit point of itself.*

THEOREM 3. *If* $2^{\aleph_1} > 2^{\aleph_0}$, *every uncountable subset of a separable normal Fréchet space-L (or -H) has a limit point.*||

* Presented to the Society, October 28, 1933.

† The number c is the power of the continuum.

‡ A space is said to be *normal* provided that, if P and Q are two mutually exclusive closed sets, there exist two mutually exclusive domains containing P and Q respectively.

§ A space is said to be *completely normal* provided that, if P and Q are two mutually separate point sets, there exist two mutually exclusive domains containing P and Q respectively.

|| The numbers $\aleph_0$ and $\aleph_1$ are the first and second transfinite cardinals respectively. That $2^{\aleph_1} > 2^{\aleph_0}$ is an immediate consequence of a well known theorem if the *hypothesis of the continuum* holds true, that is, if $\aleph_1 = c$.

THEOREM 4. *If* $2^{\aleph_1} > 2^{\aleph_0}$, *every uncountable subset of a separable completely normal Fréchet space-L (or -H) contains a limit point of itself.*

A space-L may, however, be separable and normal but contain an uncountable point set *not* containing a limit point of itself. This is shown by the following example. A lemma will be established first.

LEMMA A. *There exists on the number interval* I(0, 1) *a point set* M *of power* $\aleph_1$ *such that if* K *is a countable subset of* M, *then* K *is an inner limiting set with respect to* M.*

PROOF. Let α denote a well-ordered sequence whose elements are the points of I(0, 1). Let P_1 denote the first point of α. Let Q_2 denote an inner limiting set of I(0, 1) of measure zero containing P_1, and let P_2 denote the first point of α in $M-Q_2$. Let Q_3 denote an inner limiting set of measure zero containing $P_1+P_2+Q_2$. Let P_3 denote the first point of α in $M-Q_3, \cdots$. In general, if $\bar{z}$ is an ordinal less than ω_1 and for each ordinal z, $1<z<\bar{z}$, P_z and Q_z are defined, then $\sum(P_z+Q_z)$, $z<\bar{z}$, is of measure zero and there exists an inner limiting set $Q_{\bar{z}}$ of I(0, 1) of measure zero containing $\sum(P_z+Q_z)$, $z<\bar{z}$. Now $\mathrm{I}(0, 1)-Q_{\bar{z}}$ is of power c. Let $P_{\bar{z}}$ denote the first point of α in $\mathrm{I}(0, 1)-Q_{\bar{z}}$. Let β denote the sequence $P_1, P_2, P_3, \cdots, P_z, \cdots$ and let M denote the subset of I(0, 1) whose points are the elements of β.

It is evident from the construction that M is of power $\aleph_1$. Suppose that K is a countable subset of M. Let $P_{\bar{z}}$ denote the first point of M in β which follows K in β and let $H=\sum P_z$, $z<\bar{z}$. Then K is a subset of H, which is clearly an inner limiting set with respect to M. But $H-K$ is countable since both H and K are countable, and therefore K is an inner limiting set with respect to M.

AN EXAMPLE. Let M' denote a subset of I(0, 1) of power $\aleph_1$ such that every countable subset of M' is an inner limiting set with respect to M' and let Z' denote the set of all points (x, y) of the plane such that (1) both x and y are rational numbers, (2) $0<x<1$, and (3) $y>0$. Furthermore, let α denote a well ordered sequence of the points of M' such that if P is a point

* The symbolism G_δ is used by some to denote an *inner limiting set.*

of M', it belongs to α and is preceded in α by only a countable subset of M'. Let S denote a space consisting of the points of M' and Z' in which *sequential limit point* is defined as follows: I. A point P of Z' is the sequential limit point of a sequence of points $P_1, P_2, P_3, \cdots$ of S provided there exists a number N such that if $n > N$, then $P_n = P$. II. A point P of M' is the sequential limit point of a sequence of points $P_1, P_2, P_3, \cdots$ of Z' provided that (1) it is the sequential limit point of $P_1, P_2, P_3, \cdots$ in the plane and (2) the line PP_n approaches the normal to $I(0, 1)$ at P as n increases without limit. III. A point P of M' is the sequential limit point of a sequence of points $P_1, P_2, P_3, \cdots$ of M' provided that for each element a of α preceding P there exists an integer N such that if $n > N$, then either (1) $P_n = P$ or (2) P_n is between a and P in α; IV. In general, a sequence of points $P_1, P_2, P_3, \cdots$ of S has a sequential limit point P provided that P is by I, II, and III the sequential limit point of each of its subsequences which lie in M' or Z'. In order that it may be easier to keep in mind which limit point notion is being used, we shall adopt the convention that if H is a subset of S, H shall denote the point set as a subset of S and H' shall denote the corresponding subset of the plane.

From the definition of the above paragraph it is easy to see that S is a separable Fréchet space-L. It will now be shown that S is normal.

Suppose that H and K are two mutually exclusive closed subsets of S. If both are uncountable, then $H \cdot M$ and $K \cdot M$ are uncountable, and letting A_1 denote the first point of H in α, B_1 denote the first point of K which follows A_1 in α, A_2 denote the first point of H which follows B_1 in $\alpha, \cdots$, we see that $A_1, A_2, A_3, \cdots$ and $B_1, B_2, B_3, \cdots$ have the same sequential limit point. But since the sets are closed and mutually exclusive, this is impossible. Consequently one of the two sets is countable. We shall suppose that K denotes the countable set.

Since no point of Z is a sequential limit point of any sequence of distinct points of S, any subset of Z is a domain. Hence if H is a subset of Z, $D_H = H$ and $D_K = S - H$ are two mutually exclusive domains containing H and K respectively. Likewise if K is a subset of Z, then $D_H = S - K$ and $D_K = K$ are two mutually exclusive domains containing H and K respectively.

On the other hand, if H and K contain points of M, then for each point P of $K\cdot M$, let a denote the first point of α which precedes P in α such that no point of $H\cdot M$ is between a and P in α. Let d_P denote P together with all points of M which are between a and P in α. Then d_P is a closed domain with respect to M containing P, and $D_{1K}=\sum d_P$ is a domain with respect to M covering $K\cdot M$ and containing no point of H. Furthermore, D_{1K} is closed, for if O were a limit point of $\sum d_P$ not belonging to $\sum d_P$, then it would belong to M and be a limit point of the points P of $K\cdot M$, and hence belong to $K\cdot M$. Since D_{1K} is countable, let $D_{1K}=O_1+O_2+O_3+\cdots$. Then $D'_{1K}=O'_1+O'_2+O'_3+\cdots$ is an inner limiting set with respect to M'. It shall be assumed for convenience that D'_{1K} does not contain the end points of $I(0, 1)$. There exists a sequence of point sets $D'_1, D'_2, D'_3, \cdots$ of $I(0, 1)$ such that (1) for each n, D'_n contains D'_{n+1}, (2) the common part of $D'_1\cdot M, D'_2\cdot M, D'_3\cdot M, \cdots$ is D'_{1K}, and (3) for each n, D'_n is the sum of a set of non-overlapping segments $d'_{1n}, d'_{2n}, d'_{3n}, \cdots$. For each segment d'_{jn}, let r'_{jn} denote the interior of a regular hexagon in the plane having d_{jn} as a diameter and let $R'_n=\sum r'_{jn}$. For each n, let t'_n denote the interior of an inverted equilateral triangle lying in R'_n with its base parallel to the X-axis and lower vertex at O'_n. Let T' denote the set of all points X' and Z' such that, for some n, X' is in t'_n. Now $D_K=D_{1K}+(T-T\cdot H)$ is a domain with respect to S. Furthermore, no point of $M-D_{1K}$ is a limit point of D_K. For if P is a point of $M-D_{1K}$, P is not a limit point of D_{1K} and there exists an integer k such that R'_k does not contain P'. But no sequence of points $P'_1, P'_2, P'_3, \cdots$ lying in T' has P' as a sequential limit point in the plane such that the line $P'P'_n$ approaches the normal to the X-axis at P' since, for each n, $P'P'_n$ would make an angle of at least 30° with this normal when P'_n lies in R'_k. Hence D_K is closed and contains K but no points of H. Therefore, $D_H=S-D_K$ and D_K are mutually exclusive domains containing H and K respectively, and S is normal.

The reader will observe that if N is an uncountable subset of S, then $N\cdot M$ is uncountable and N has a limit point, namely, the first point P of α such that infinitely many points of M precede P in α. But it is clear that not every uncountable subset of S *contains* one of its limit points; for suppose that N is the set of all points P of M such that there is a first point of M in α preceding P in α. Then N is uncountable and contains none of its limit points.

In order to make an application of Theorem 4, two lemmas will be established. Throughout the rest of this paper M denotes a space satisfying Axiom 0 and parts 1, 2, and 3 of Axiom 1 of R. L. Moore's *Foundations of Point Set Theory* and is referred to as a *Moore space M.*

DEFINITION. A space is said to have the *Lindelöf property* provided that if G is a collection of domains of the space covering a point set K, then G contains a countable subcollection G' covering K.

LEMMA B. *In order that a Moore space M should have the Lindelöf property it is necessary and sufficient that every uncountable subset of M should have a limit point.**

The necessity is well known. It remains only to establish the sufficiency.

PROOF. Suppose that G is a collection of domains covering a point set K. Let α denote a well-ordering of K. For each n, let H_n denote a subcollection of G obtained by the following method. Let P_1 denote the first element of α such that some element g_1 of G contains every region of G_n of Axiom 1 that contains P_1. Let P_2 denote the first element of α, not contained in g_1, such that some element g_2 of G contains every region of G_n of Axiom 1 that contains P_2. In general, if $\bar{z}$ is an ordinal and for each ordinal z, $z<\bar{z}$, P_z and g_z are chosen, then let $P_{\bar{z}}$ denote the first point (if any) in α not contained in $\sum g_z$, $z<\bar{z}$, such that some element $g_{\bar{z}}$ of G contains every region of G_n of Axiom 1 that contains $P_{\bar{z}}$. From this construction, it is clear that the set $P_1, P_2, P_3, \cdots, P_z, \cdots$ has no limit point, for no region of G_n contains more than one of them. Hence $H_n = g_1, g_2, g_3, \cdots, g_z, \cdots$ is a countable subcollection of G. Then $G' = \sum_1^\infty H_n$ is a countable subcollection of G. Furthermore, G' covers K. For suppose that there is a point P of K not contained in any element of G'. Let g denote a domain of G containing P. By

* Lemma B is an advance over Theorem 18 on page 14 of R. L. Moore's *Foundations of Point Set Theory*. However, Moore's arguments may be used with some modifications to establish Lemma B.

Axiom 1 there exists a number n such that every region of G_n which contains P lies in g. Hence P, for some ordinal $\bar{z}$, is $P_{\bar{z}}$ used in the selection of H_n, which is a contradiction.

LEMMA C. *If every uncountable subset of a Moore space M has a limit point, M is a completely separable metric space.*

PROOF. By Lemma B, for each n, G_n of Axiom 1 contains a countable subcollection G_n' covering M; hence $G' = \sum G_n'$ is a countable collection of regions having the property that if P is a point of a region R, some element of G' contains P and lies in R. Hence M is completely separable. Professor Moore has pointed out that such a space is metric.*

THEOREM 5. *If $2^{\aleph_1} > 2^{\aleph_0}$, then every separable normal Moore space M is completely separable and metric.*

This follows from Theorem 4 and Lemma C.

The author has tried for some time without success to prove that $2^{\aleph_1} > 2^{\aleph_0}$. But although Theorem 5 is unsatisfactory in this respect, it does raise a question of some interest: *Is every normal Moore space M metric*? This question is as yet unsettled. However, if the answer is *yes*, then it should be possible to establish directly certain results for normal Moore spaces M which are known to hold in metric spaces but which are known not to hold in all Moore spaces M. The author has established a number of such theorems but it seems likely that only one of them may be of use in settling the question itself.

THEOREM 6. *A normal Moore space M is completely normal.*

PROOF. Suppose that H and K are two mutually separate subsets of a normal Moore space M. For each integer n, let H_n denote the set of all points P of $\overline{H}$ such that no region of G_n of Axiom 1 which contains P contains a point of $\overline{K}$.† Likewise, for each n, let K_n denote the set of all points P of $\overline{K}$ such that no region of G_n which contains P contains a point of $\overline{H}$. For each n, H_n and K_n are closed and $H \subset \sum H_n$ and $K \subset \sum K_n$. Let D_{H_1} denote a domain containing H_1 such that $\overline{D}_{H_1} \cdot \overline{K} = 0$. Let D_{K_1} denote a domain containing K_1 such that $\overline{D}_{K_1} \cdot \overline{(H + D_{H_1})} = 0$.

* R. L. Moore, *Foundations of Point Set Theory*, pp. 459 and 464.

† The notation $\overline{K}$ means K plus its limit points.

Let D_{H_2} denote a domain containing H_2 such that $\overline{D}_{H_2} \cdot \overline{(K+D_{K_1})} = 0$. Let D_{K_2} denote a domain containing K_2 and such that $\overline{D}_{K_2} \cdot \overline{(H+D_{H_1}+D_{H_2})} = 0$. This process may be continued and $D_H = \sum D_{H_n}$ and $D_K = \sum D_{K_n}$ are two mutually exclusive domains covering H and K respectively.

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ON AN INTEGRAL EQUATION WITH AN ALMOST PERIODIC SOLUTION

BY B. LEWITAN

We assume that the function $f(x)$ is almost periodic in the sense of H. Bohr and that the functions $E(\alpha)$, $\alpha E(\alpha)$ are absolutely integrable in $[-\infty, \infty]$.

THEOREM. *If all real zeros of the function*

$$\gamma(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(u) e^{-i\alpha u} du$$

have integer multiplicities and only two limit points ∞, α^*, *then every solution* $\phi(x)$ *of the equation*

$$\int_{-\infty}^{\infty} E(\xi - x) \cdot \phi(\xi) d\xi = f(x) \tag{1}$$

which is uniformly continuous and bounded in $[-\infty, \infty]$ *is almost periodic.*

PROOF. Without loss of generality we may assume that the finite limit point α^* has the value 0; otherwise we multiply equation (1) by $e^{-i\alpha^* x}$.

Putting

$$f_n(x) = \frac{3}{2\pi} \int_{-\infty}^{\infty} f\left(x + \frac{2u}{n}\right) \frac{\sin^4 u}{u^4} du,$$

we obtain

$$\int_{-\infty}^{\infty} E(\xi) \phi_n(\xi + x) d\xi = f_n(x),$$