DE DONDER ON CALCULUS OF VARIATIONS

Théorie Invariantive du Calcul des Variations. By Th. de Donder. Paris, Gauthier-Villars, 1935. x+230 pp.

This book develops the purely formal parts of the calculus of variations. The title is justified by the study of the effect of transformations of coordinates and parameters upon the fundamental integral (1) and upon other expressions which occur in the theory. The author especially examines the conditions for the invariance of the integral under coordinate transformations.

The book is divided into three parts. The first of these consists of seven chapters, which contain the following material. Let V_τ be a piece of Euclidean $(x^1, \cdots, x^n)$-space depending on a parameter τ. Consider, furthermore, functions $y^i(x, \tau)$, $i=1, \cdots, m$, and a function F depending on τ, the x^i, the y^i, and their partial derivatives up to a certain order c, and form the integral

$$(1)\qquad I(\tau) = \int_{V_\tau} F\left(x^i, y^i, \frac{\partial y^i}{\partial x^j}, \frac{\partial^2 y^i}{\partial x^h \partial x^j}, \cdots, \tau\right) dx^1 \cdots dx^n.$$

Then $I'(\tau)$ is expressed as the sum of an integral over V_τ and an integral over the boundary of V_τ. The method is extended to the case where V_τ is a piece of a surface of an arbitrary dimension in the n-space. The Stokes formula in a very general form and other classical results are derived as applications. Then the higher derivatives of $I(\tau)$ are represented in various forms. Five different expressions are given for $I''(\tau)$. These results are generalized by making the deformation of V depend on several parameters $\tau_1, \cdots, \tau_\sigma$ instead of one, yielding formulas for $\partial I(\tau_1, \cdots, \tau_\sigma)/\partial \tau_i$ and $\partial^2 I(\tau_1, \cdots, \tau_\sigma)/\partial \tau_i \partial \tau_j$. Chapter IV deals with integrals in the parametric form, that is, integrals $I(\tau)$ and $I(\tau_1, \cdots, \tau_\sigma)$ which are invariant under change of the variables $x^1, \cdots, x^n$ into any curvilinear coordinates. For $I(\tau_1, \cdots, \tau_\sigma)$ one finds, besides the usual conditions, additional ones that are rather complicated. The piece V_τ (dim $V_\tau < n$) is next represented implicitly as the intersection of manifolds $\Phi(x^1, \cdots, x^n) = 0$ or pieces thereof, and again conditions are given under which the integral is independent of the choice of the coordinates. The last chapter of the first part examines thoroughly the behavior of the expressions previously considered under transformations of the x^i and shows that the methods developed here contain the main formulas of the absolute differential calculus as special cases.

The second part deals more properly with the calculus of variations. It starts with the definition of extremals for the parametric and the non-parametric cases, gives their invariant properties, and treats the theories of Lagrange and Hamilton-Jacobi. It is then occupied with the theorem of Jacobi concerning the relations between the Hamiltonian function and the integration of the canonical equations of a variation problem. First the author disposes of the classical case in which only one independent variable and only the first partial derivatives occur. Then generalizations in both directions are obtained, the calculations for the most general case becoming very complicated. In the same way the classical theory of the Hilbert integral and the Weierstrass excess formula are treated. The end of the second part treats the equations of variation and generalizes H. Hahn's research on the connections between the theory of the second variation and the Weierstrass theory.

The third part contains applications to mathematical physics. It will be sufficient to name the subjects touched upon: the theory of gravitation, an electrically charged point mass in a gravitational and electrical field, the wave theory of light, and wave mechanics. Then the considerations again become purely mathematical for a time. That is, the theory of adjoint and self-adjoint systems of differential equations is briefly surveyed and applied to physical problems. The final chapter treats more closely the case where $F(x, y, \cdots, \tau)$ depends neither on τ nor on derivatives of higher order than the first, and applies these results to some problems of capillarity.

The reader of this review will miss some indications regarding the assumptions under which all these results hold. So did the reviewer when reading the book. He felt himself carried back to the heroic age of mathematics, when doubts had not yet retarded our steps. How smoothly runs the author's proof of the most general Stokes formula compared with the troubles which meager special cases cause to others! (The author actually compares his quite uncritical method with that in the book of Courant and Hilbert, p. 19.) The whole book can be regarded only as a collection of purely formal relations which hold locally under sufficiently strong conditions of regularity. I mention as an illustration that no actual use is ever made of conjugate points or of the concept of a field of extremals (the word is mentioned). Sometimes even the formal assumptions must be completed. For instance, §46, especially formula (543), could not be obtained if F really depended on τ, a possibility which the author did not exclude. One can see to what extent formalism prevails when the author says (p. 97), "on prend *par convention* $\delta' x = 0$," and means that the end points of the extremal under consideration are moved *transversely*.

Nevertheless, since one aspect of the calculus of variations is treated so completely, the book will be of use for some questions of research. For instance, one finds many interesting identities.

There seem to be very few misprints and only unessential ones (for instance in formula (685)). This is astonishing in view of the numerous complicated formulas. The print is excellent. One would recommend a more extended use of the summation convention of the tensor calculus; and a list of the many symbols would be helpful.

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