

HADAMARD'S MEMORIAL VOLUME

Selecta. Jubilé Scientifique de M. Jacques Hadamard. Paris, Gauthier-Villars, 1935. 432 pp.

The task of selecting a group of representative papers from the writings of a mathematician as prolific and versatile as Hadamard is not an easy one. Obviously, papers containing fundamental results which have attracted almost universal attention should be included. On the other hand, a certain unity, showing a concept at different stages of development, is also desirable if such a volume is to prove useful as well as ornamental. It seems to the reviewer that the editors of this volume have succeeded admirably in their task, and have produced a book which not only gives an insight into the mathematical development of the author, but is also valuable as a source book and reference volume.

Since most of the papers have been in print for some time, only a brief survey of the volume is necessary. The papers have been roughly grouped under various headings, which give an indication of the variety of interests of the author, and his ability to make contributions in divergent fields. The first section is devoted to the Theory of Analytic Functions, and the first paper is, of course, his Paris doctoral thesis, entitled *Essai sur l'étude des fonctions données par leur développement de Taylor* (1892). It contains the introduction of the notion of greatest and least of the limits of a sequence utilized in the theorem now known as the Cauchy-Hadamard theorem for the radius of convergence of a power series, which is then applied to the study of conditions for the existence of a finite number of polar singularities on the circle of convergence. The thesis also includes sufficient conditions that the circle of convergence be a line of singularities. The third part of the thesis is not included. The second paper (1893) in this group, on entire functions, contains extensions of Poincaré's results on the relations between the growth of the coefficients, the distribution of zeros, and the order of entire functions. Of the three remaining papers in this set, mention might be made of the paper (1898) giving the relations of the singularities of the function $h(x) = \sum c_n x^n$ to those of $f(x) = \sum a_n x^n$ and $g(x) = \sum b_n x^n$ when $c_n = a_n b_n$. The last paper of this series, a meditation on the generalization of analytic functions (1912), anticipates the type of study of derivatives of functions of real variables fundamental to the theory of quasi-analytic functions.

The section devoted to the Theory of Numbers is number-theoretic mainly by implication. It centers in two studies of the Riemann zeta function, the first of which (1896) demonstrates that $\zeta(s)$ has no zeros on the line $\Re(s) = 1$, and shows that the same methods are applicable to certain Dirichlet series. The second (1928) applies the formula generalizing Parseval's theorem relative to two developments in Dirichlet series to the function $\zeta'(s)/\zeta(s)$. The third paper in this set is the famous paper on the maximum value of a determinant (1893).

The next section headed Real Functions contains a single paper, *Sur les*

transformations ponctuelles (1906), concerned with the question of inversion of the system $X_i = f_i(x_1, \dots, x_k)$, ($i = 1, \dots, k$), x and X ranging over euclidean space of k dimensions, which calls attention to the need of conditions sufficient to insure not only the uniqueness, but the existence of an inverse.

In the division entitled *Differential Equations and Partial Differential Equations*, the first paper is really concerned with a study of what happens to a given curve near the origin under iteration of a system of functional equations $x_1 = f(x, y) = ax + by + \dots$, $y_1 = g(x, y) = cx + dy + \dots$, the roots of the characteristic equation $(a-s)(d-s) - bc = 0$ being real and distinct. This paper is connected with differential equations only in that this problem arises in a study of Poincaré on the periodic solutions of differential equations. The next paper, *Sur les surfaces à courbures opposées et leurs lignes géodésiques* (1898), contains his classic results concerning geodesics on surfaces of this type without singularities in finite euclidean space and extending necessarily to infinity. He applies the Poincaré analysis situs in order to obtain a relation between the connectivity of the surface and the number of closed trajectories, probably the earliest suggestion of the effectiveness of analysis situs methods in the calculus of variations. The other two papers in this section devoted to partial differential equations center in the contrast between the Cauchy or initial condition problem, and the Dirichlet or boundary value problem, the first suggesting that a "sensible" question in mathematical physics leads to a problem with possibility of solution, the second paper pointing out that the statements of existence of solutions for the Cauchy type of problem must be examined carefully; for example, in the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial t^2}$$

the Cauchy problem is possible of solution if t is the principal variable, but not always when x , y , or z is the principal variable.

In the part devoted to Calculus of Variations and the Calcul Fonctionnel, one finds, of course, the paper giving the first solution of the problem of determining the most general linear operation in the space of continuous functions on a finite range, superseded to be sure by Riesz's beautiful result, but still suggestive. The remaining papers include an outline of a direct method for proving the existence of a minimum for the simplest problem of the calculus of variations, a study of the stationary point of a function $f(x, y, z)$ on a curve $g_1(x, y, z) = g_2(x, y, z) = 0$, and the determination of some special solutions of the functional equation

$$\frac{\partial}{\partial \alpha} \psi(A, B, \alpha) = - \int \nu(s, \alpha) \psi(A, M, \alpha) \psi(B, M, \alpha) ds$$

associated with Green's function.

The Geometry section includes two papers of an elementary character, one a note from his *Leçons de Géométrie* (1898), giving good sound advice on the analysis and solutions of geometric problems and demonstrations, and a paper showing how to determine directly the double points of a curve given in implicit parameter form, $f_1(x, y, t) = f_2(x, y, t) = 0$. There is also the note from Tannery's book on function theory, in which the Kronecker index is used to

deduce from the Jordan curve theorem the Schoenflies theorem that if the continuous single-valued transformation $X=f(x, y)$, $Y=g(x, y)$, of the circle $x^2+y^2=1$ satisfies the condition that $f(x, y)=f(x', y')$, $g(x, y)=g(x', y')$ implies $(x, y)=(x', y')$ in and on this circle, then the transformation of the interior of the curve C corresponding to $x^2+y^2=1$ is one to one, and links it with the Brouwer fixed point theorem. The closing paper is devoted to a portion of géométrie anallagmatique, the geometry associated with inversions relative to a sphere.

The final section containing the most recent papers is devoted to the subject of Hydrodynamics, and most of these papers center in questions growing out of the principle of Huyghens stated in the form "from the state of the universe at the instant t_0 one can deduce the state at a later instant t' , the deduction being possible by passing from t_0 to t_1 (t_1 being any instant such that $t_0 < t_1 < t'$) and from t_1 to t' ." The papers are concerned largely with questions relating to the solutions of partial differential equations of the second order with boundary conditions of the mixed type, that is, conditions of initial or Cauchy type relative to one surface S , and of Dirichlet type along another surface Σ . The second paper is largely geometrical, centering in the contact between characteristics and the caustic curves derived from a system $z=f(x, y, \alpha, \beta)$ with two parameters. These results are applied in the third paper to study the solutions of partial differential equations.

A list of Hadamard's publications by years brings the volume to a close. The book as a whole leaves the impression that the editors have succeeded in producing a memorial volume worthy of the name, giving on the one hand an insight into the development of Hadamard's mathematical ideas, and, on the other, collecting in convenient form a group of papers which have played such an important role in the development of many phases of mathematics in the last forty years. In addition to their mathematical value, many of the papers of this group are models of clear exposition, worthy of emulation.

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