tetrahedron outside its edges each in a *Caporali quartic, and contains both the* G_{27} *and the* G_{36}.

$$
\begin{aligned}
F_7 = x_1x_2x_3\{&a_{14}x_1(x_2^3 - x_3^3) + a_{24}x_2(x_3^3 - x_1^3) + a_{34}x_3(x_1^3 - x_2^3)\} \\
+ x_1x_2x_4\{&a_{13}x_1(x_2^3 - x_4^3) + a_{23}x_2(x_4^3 - x_1^3) + a_{43}x_4(x_1^3 - x_2^3)\} \\
+ x_1x_3x_4\{&a_{12}x_1(x_3^3 - x_4^3) + a_{32}x_3(x_4^3 - x_1^3) + a_{42}x_4(x_1^3 - x_3^3)\} \\
+ x_2x_3x_4\{&a_{21}x_2(x_3^3 - x_4^3) + a_{31}x_3(x_4^3 - x_2^3) + a_{41}x_4(x_2^3 - x_3^3)\} = 0.
\end{aligned}
$$

It has the A_i*'s as triple points and the* $\overline{A_iA_k}$ *as single lines.*

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EINSTEIN SPACES OF CLASS ONE*

BY C. B. ALLENDOERFER

1. *Introduction.* An Einstein space is defined as a Riemann space for which

$$(1) \qquad R_{\alpha\beta} = \frac{R}{n} g_{\alpha\beta}.$$

We assume the first fundamental form

$$(2) \qquad ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$$

to be non-singular, but do not restrict ourselves to the positive definite case. An $n+1$ dimensional space is said to be flat when its first fundamental form can be reduced to†

$$(3) \qquad ds^2 = \sum_{i=1}^{n+1} c_i(dx^i)^2,$$

where the c_i are definitely plus one or minus one. An n dimensional Riemann space which is not flat is said to be of class one if it can be imbedded in an $n+1$ dimensional flat space. The purpose of this paper is to determine necessary and sufficient conditions that an Einstein space be of class one.

There is no problem when $n=2$, for then every space which

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† Throughout this paper Latin indices will have the range 1 to $n+1$; Greek indices the range 1 to n.

is not flat is of class one. Neither need we consider the case $n=3$, for then every Einstein space is a space of constant curvature and consequently is of class one unless it is flat.* It has further been shown that there can exist no Einstein space with a positive definite form (2) which is of class one and for which $R=0$.† There may, however, exist spaces of class one with $R=0$ when the form (2) is indefinite. Brinkmann‡ has shown that this property is enjoyed for $n=4$ by a space with the first fundamental form

$$ds^2 = 2dxdy + 2d\phi d\theta + 2f(x, \phi)dxd\phi,$$

where $f(x, \phi)$ is any function possessing continuous first and second derivatives. In this paper we consider the case where $n>3$ and $R\neq 0$, thus leaving open the case where $R=0$ and the form (2) is indefinite. Our procedure will consist of reducing the problem to a purely algebraic one by means of a recent theorem of T. Y. Thomas;§ and then the algebraic problem is solved by straightforward elimination methods.

2. *Riemann Spaces of Class One.* Before discussing Einstein spaces, let us recall the general theory of a Riemann space of class one. Let U be a simply connected neighborhood of the n dimensional arithmetic number space. A Riemann space will be called an R_2 in U if the coefficients of the first fundamental form (2) are functions of class C^2 for $x \subset U$ and the components of the Riemann curvature tensor are of class C^1 for $x \subset U$. An R_2 is said to be imbedded in the $n+1$ dimensional flat space whose metric is given by (3) when there exist functions $y^i = \phi^i(x)$ of class C^2 in U which satisfy the mixed system

* J. A. Schouten and D. J. Struik, *On some properties of general manifolds relating to Einstein's theory of gravitation*, American Journal of Mathematics, vol. 43 (1921), p. 214.

† E. Kasner, *The impossibility of Einstein fields immersed in flat space of five dimensions*, American Journal of Mathematics, vol. 43 (1921), p. 126. See also L. P. Eisenhart, *Riemannian Geometry*, 1926, for a discussion of Einstein spaces and for the general theory of spaces of class one on which this discussion is based.

‡ H. W. Brinkmann, *Conformal mapping of Einstein spaces*, Mathematische Annalen, vol. 94 (1925), pp. 140–141.

§ T. Y. Thomas, *Riemann spaces of class one and their characterization*, Acta Mathematica, vol. 67 (1936), pp. 169–211.

$$\frac{\partial y^i}{\partial x^\alpha} = y^i_{,\alpha},$$

$$(4) \qquad \frac{\partial(y^i_{,\alpha})}{\partial x^\beta} = \left\{ \begin{matrix} \gamma \\ \alpha\beta \end{matrix} \right\} y^i_{,\gamma} + e b_{\alpha\beta}\sigma^i,$$

$$\frac{\partial(\sigma^i)}{\partial x^\alpha} = - b_{\alpha\beta} g^{\beta\gamma} y^i_{,\gamma},$$

and

$$(5) \qquad \sum_{i=1}^{n+1} c_i y^i_{,\alpha} y^i_{,\beta} = g_{\alpha\beta}, \qquad \sum_{i=1}^{n+1} c_i y^i_{,\alpha}\sigma^i = 0, \qquad \sum_{i=1}^{n+1} c_i \sigma^i \sigma^i = e,$$

where σ^i is the unit normal to R_2, $b_{\alpha\beta}$ are the coefficients of the second fundamental form of R_2, and e is definitely plus one or minus one. The functions $b_{\alpha\beta}(x)$ and σ^i are of class C^1 in U. This system is completely integrable if and only if the Gauss equations

$$(6) \qquad R_{\alpha\beta\mu\rho} = e(b_{\alpha\mu}b_{\beta\rho} - b_{\alpha\rho}b_{\beta\mu}),$$

and the Codazzi equations

$$(7) \qquad b_{\alpha\mu,\rho} = b_{\alpha\rho,\mu}$$

are satisfied for all $x \subset U$. Under these circumstances the system has a unique solution in U for every set of initial values $(y^i)_0$, $(y^i_{,\alpha})_0$, and $(\sigma^i)_0$ which satisfy (5) at the point P where $x = x_0$. *A given R_2 defined in a neighborhood U is of class one if and only if its curvature tensor does not vanish and there can be found a value of e and a set of functions $b_{\alpha\beta}(x)$ of class C^1 which satisfy* (6) *and* (7) *in U.*

Suppose now that we are given a Riemann space whose first fundamental form is of signature s, and the coefficients of whose second fundamental form satisfy (6) and (7) for a particular value of e. We seek to determine the signature of the flat space in which it can be imbedded. So we choose a coordinate system such that at the initial point, P, $g_{\alpha\beta} = 0$ for $\alpha \neq \beta$, $g_{\alpha\alpha} = +1$ for $\alpha = 1, \cdots, (s+n)/2$, and $g_{\alpha\alpha} = -1$ for $\alpha = [(s+n)/2]+1, \cdots, n$. An obvious set of c_i and initial values of the variables which satisfies (5) at P is obtained by putting $c_i = +1$ for $i = 1, \cdots, (s+n)/2$, $c_i = -1$ for $i = [(s+n)/2]+1, \cdots, n$; $c_{n+1} = e$; and $(y^i_{,\alpha})_0 = \delta^i_\alpha$; $(\sigma^i)_0 = 0$ for $i = 1, \cdots, n$; $(\sigma^{n+1})_0 = 1$.

The signature of the flat space so determined is thus $s+e$. The usual argument* shows that this signature is the same for all sets of solutions of (5).

3. *Einstein Spaces of Class One.* We now pass to the consideration of Einstein spaces and suppose that we are given an Einstein space of class one which is furthermore an R_2 in U. Then there exist real functions $b_{\alpha\beta}(x)$ of class C^1 and a value of e which satisfy (6). In order to solve (6) for the $b_{\alpha\beta}(x)$, we observe that from (6) we have

$$\begin{aligned}(8)\qquad & 2e(b_{\gamma\tau}b_{\beta\mu}R_{\alpha\delta\sigma\rho} + b_{\gamma\rho}b_{\beta\mu}R_{\alpha\delta\tau\sigma} + b_{\gamma\sigma}b_{\beta\mu}R_{\alpha\delta\rho\tau}) \\ &\quad = R_{\alpha\beta\mu\sigma}R_{\gamma\delta\rho\tau} + R_{\alpha\beta\mu\tau}R_{\gamma\delta\sigma\rho} + R_{\alpha\beta\mu\rho}R_{\gamma\delta\tau\sigma} - R_{\delta\beta\mu\sigma}R_{\gamma\alpha\rho\tau} \\ &\quad - R_{\delta\beta\mu\tau}R_{\gamma\alpha\sigma\rho} - R_{\delta\beta\mu\rho}R_{\gamma\alpha\tau\sigma} - R_{\alpha\delta\mu\tau}R_{\gamma\beta\rho\sigma} \\ &\quad - R_{\alpha\delta\mu\rho}R_{\gamma\beta\sigma\tau} - R_{\alpha\delta\mu\sigma}R_{\gamma\beta\tau\rho}.\end{aligned}$$

Multiplying (8) by $g^{\alpha\sigma}g^{\delta\rho}$, summing, and using (1), we find that

$$\begin{aligned}(9)\qquad & b_{\gamma\tau}b_{\beta\mu} \\ &= e\left[\frac{R}{n(2-n)}g_{\gamma\tau}g_{\beta\mu} - \frac{n}{2R(2-n)}g^{\alpha\sigma}g^{\delta\rho}(2R_{\alpha\beta\mu\rho}R_{\delta\gamma\tau\sigma} + R_{\alpha\delta\mu\tau}R_{\gamma\beta\rho\sigma})\right] \\ &= e\,D_{\gamma\tau|\beta\mu},\end{aligned}$$

where the $D_{\gamma\tau|\beta\mu}$ are thus defined. It follows at once that

$$(10)\qquad b_{\gamma\tau} = \pm\,(eD_{\gamma\tau|\gamma\tau})^{1/2}.$$

Since $b_{\gamma\tau}$ are real, *the matrix*

$$M \equiv \| D_{\gamma\tau|\beta\mu} \|$$

is semi-definite of rank one in U, *where* $(\gamma\tau)$ *indicates the row and* $(\beta\mu)$ *the column.* We call this property condition A.

Interchanging τ and μ in (9), subtracting the result from (9), and using (6), we find that, for every point in U,

$$\begin{aligned}(11)\qquad R_{\gamma\beta\tau\mu} &= \frac{R}{n(2-n)}(g_{\gamma\tau}g_{\beta\mu} - g_{\gamma\mu}g_{\beta\tau}) \\ &\quad - \frac{n}{R(2-n)}g^{\alpha\sigma}g^{\delta\rho}(R_{\alpha\beta\mu\rho}R_{\delta\gamma\tau\sigma} - R_{\alpha\gamma\mu\rho}R_{\delta\beta\tau\sigma} + R_{\alpha\delta\mu\tau}R_{\gamma\beta\rho\sigma}).\end{aligned}$$

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* See, for example, Duschek-Mayer, *Lehrbuch der Differentialgeometrie*, 1930, vol. 2, p. 26.

We call this condition B, thus having proved that conditions A and B are necessary that the given space be of class one.

To show that they are sufficient, suppose an Einstein space ($R\neq 0$) is given which satisfies A and B and which is also an R_2 in U. Since M is semi-definite of rank one in U, we know that at any point $P\subset U$ there exist a unique determination of e and real values of $b_{\alpha\beta}$ which are unique to within algebraic sign, such that (9) is satisfied. At least one of these, say $b_{\xi\eta}$, is not zero at P, and its value is given by

$$b_{\xi\eta} = \pm (eD_{\xi\eta|\xi\eta})^{1/2}, \tag{12}$$

where $D_{\xi\eta|\xi\eta}\neq 0$ at P and e is chosen such that $b_{\xi\eta}$ is real. The values of the $b_{\beta\mu}$ for other sets of indices may be obtained from

$$b_{\beta\mu} = \frac{eD_{\beta\mu|\xi\eta}}{b_{\xi\eta}}. \tag{13}$$

Since $D_{\xi\eta|\xi\eta}$ is of class C^1 in U, there exists a neighborhood $V(P)$ such that $P\subset V(P)\subset U$ within which $D_{\xi\eta|\xi\eta}\neq 0$ and $b_{\xi\eta}\neq 0$. Since the other $D_{\beta\mu|\xi\eta}$ are also of class C^1 in U, it follows that for each choice of the sign in (12), equations (12) and (13) define a constant value of e and a set of real $b_{\alpha\beta}(x)$ which are of class C^1 in $V(P)$. It is clear that the choice of sign throughout $V(P)$ is determined by its value at any point. These $b_{\alpha\beta}(x)$ and e satisfy (9) by their very definition, and since (11) is satisfied in U by hypothesis, they also satisfy (6) in $V(P)$.

Now consider a point $Q\subset V(P)\subset V(P')$, where P and P' are distinct points. By the above procedure define a definite set of $b_{\alpha\beta}(Q)$ and a value of e, say $e(Q)$, at Q, when Q is considered as an element of $V(P)$. Another set, $b'_{\alpha\beta}(Q)$ and $e'(Q)$, are defined when Q is considered as an element of $V(P')$. But since the value of e is unique at any point and is constant in any V neighborhood, it follows that e is constant throughout $V(P)+V(P')$. And since the possible values of $b_{\alpha\beta}$ at any point are unique to within algebraic sign, it is clear that the sign to be chosen in (12) for $V(P')$ can be taken such that $b_{\alpha\beta}(Q)=b'_{\alpha\beta}(Q)$. Since $b_{\alpha\beta}(x)$ and $b'_{\alpha\beta}(x)$ are of class C^1 in $V(P)$ and $V(P')$, respectively, it follows that $b_{\alpha\beta}(x)=b'_{\alpha\beta}(x)$ throughout $V(P)\cap V(P')$. Thus we have defined a solution of (6) which is of class C^1 in $V(P)+V(P')$.

Now T. Y. Thomas* has shown that this method can be used to extend this solution to the entire neighborhood U. The result is that A *and* B *are sufficient conditions for the existence of a set of* $b_{\alpha\beta}(x)$ *of class* C^1 *and a constant value of e which satisfy* (6) *in* U.

From this we can show at once that the $b_{\alpha\beta}(x)$ also satisfy (7) in U. For, first, the matrix $\|b_{\alpha\beta}\|$ is of rank n in U. Multiply (6) by $g^{\alpha\rho}$, sum, and make use of (1). There results

$$g_{\beta\mu}\frac{R}{n} = eg^{\alpha\rho}b_{\alpha\mu}b_{\beta\rho} - b_{\alpha\rho}g^{\alpha\rho}b_{\beta\mu}. \tag{14}$$

Now if $\|b_{\alpha\beta}\|$ were of rank $<n$ at any point $P \subset U$, the system $b_{\alpha\mu}\lambda^\mu = 0$ would have at least one non-zero solution at P. But if this were the case, (14) would show that $g_{\alpha\mu}\lambda^\mu = 0$ has at least one non-zero solution at P, but this is impossible, since we have assumed that $|g_{\alpha\beta}| \neq 0$ in U. Since T. Y. Thomas has shown in the paper mentioned above that if the rank of $\|b_{\alpha\beta}\|$ is >3, the Codazzi equations are satisfied as consequences of the Gauss equations, it follows that our $b_{\alpha\beta}(x)$ satisfy (7) in U.

THEOREM. *An Einstein space with* $n>3$ *and* $R\neq 0$ *which is an* R_2 *in a simply connected neighborhood* U *is a space of class one if and only if* $\|D_{\gamma\tau|\beta\mu}\|$ *is semi-definite of rank one in* U, *and equations* (11) *are satisfied in* U. *If s is the signature of the first fundamental form of an Einstein space which satisfies the above conditions, the space can be imbedded in any of the flat spaces whose first fundamental forms are of signature* $s+e$, *where e is determined as in* (12).

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* T. Y. Thomas, *Riemann spaces of class one and their characterization*, Acta Mathematica, vol. 67 (1936), p. 205.