ON CERTAIN CONFIGURATIONS OF POINTS IN SPACE AND LINEAR SYSTEMS OF SURFACES WITH THESE AS BASE POINTS*

BY ARNOLD EMCH

1. *Introduction.* Configurations of this sort in connection with certain surfaces are known in large numbers. For example, the vertices of the 45 triangles formed by the 27 lines on a general cubic surface; the 12 vertices of 3 desmic tetrahedra; the 24 double points of the 6 quintic cycles of the symmetric collineation group on five variables interpreted in S_3; the G_{18} group of points which I found on a new normal form of the cubic surface,† and so on.

In this paper I shall establish two new configurations of points and investigate their properties and some of the surfaces on these points.

2. *The G_{27} of W-Points.* This configuration is defined by the system of points W

$$(1)\quad W = (\omega^\alpha, \omega^\beta, \omega^\gamma, 1), \quad \omega^3 = 1, \quad \alpha, \beta, \gamma \equiv 0, 1, 2, (\text{mod}\, 3),$$

which yields the group G_{27} of 27 points W. Consider now any of the W's and two more of the set as follows:

$$\begin{aligned} W_0 &= (\omega^\alpha, \quad \omega^\beta, \quad \omega^\gamma, \quad 1), \\ W_1 &= (\omega^{\alpha+1}, \omega^{\beta+1}, \omega^{\gamma+1}, 1), \\ W_2 &= (\omega^{\alpha+2}, \omega^{\beta+2}, \omega^{\gamma+2}, 1). \end{aligned}$$

Subtracting corresponding coordinates of these three points, say $(W_0 - W_1)$, $(W_1 - W_2)$, $(W_2 - W_0)$, and dividing in each case by $(1-\omega)$, we obtain the point $V(\omega^\alpha, \omega^\beta, \omega^\gamma, 0)$. The cross-ratio of the four points is

$$(VW_0W_1W_2) = (\infty, \omega^\alpha, \omega^{\alpha+1}, \omega^{\alpha+2}) = (\infty, 1, \omega, \omega^2) = -\omega^2.$$

THEOREM 1. *Every V-point is collinear with three W-points. The cross-ratio of these four points is equianharmonic.*

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† American Journal of Mathematics, vol. 53 (1931), pp. 902–910.

Next, consider any of the A_i, say $A_1(1, 0, 0, 0)$, and the triples of points

$$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ \omega & 1 & 1 & 1 \\ \omega^2 & 1 & 1 & 1 \end{array} \qquad \begin{array}{cccc} 1 & \omega & \omega & 1 \\ \omega & \omega & \omega & 1 \\ \omega^2 & \omega & \omega & 1 \end{array} \qquad \begin{array}{cccc} 1 & \omega^2 & \omega^2 & 1 \\ \omega & \omega^2 & \omega^2 & 1 \\ \omega^2 & \omega^2 & \omega^2 & 1 \end{array} \qquad \begin{array}{cccc} 1 & \omega & 1 & 1 \\ \omega & \omega & 1 & 1 \\ \omega^2 & \omega & 1 & 1 \end{array} \qquad \begin{array}{cccc} 1 & 1 & \omega & 1 \\ \omega & 1 & \omega & 1 \\ \omega^2 & 1 & \omega & 1 \end{array}$$

$$\begin{array}{cccc} 1 & \omega^2 & 1 & 1 \\ \omega & \omega^2 & 1 & 1 \\ \omega^2 & \omega^2 & 1 & 1 \end{array} \qquad \begin{array}{cccc} 1 & 1 & \omega^2 & 1 \\ \omega & 1 & \omega^2 & 1 \\ \omega^2 & 1 & \omega^2 & 1 \end{array} \qquad \begin{array}{cccc} 1 & \omega & \omega^2 & 1 \\ \omega & \omega & \omega^2 & 1 \\ \omega^2 & \omega & \omega^2 & 1 \end{array} \qquad \begin{array}{cccc} 1 & \omega^2 & \omega & 1 \\ \omega & \omega^2 & \omega & 1 \\ \omega^2 & \omega^2 & \omega & 1 \end{array}$$

The points of each of these triples are collinear with A_1. Moreover the cross-ratio of A_1 and any of the triples is $(\infty, 1, \omega, \omega^2) = -\omega^2$. Taking account of the symmetry of the G_{27} we may state the following theorem.

THEOREM 2. *The* 27 *W-points lie* 4 *times on* 9 *lines through the* A_i*'s, each line containing* 3 *of the W's. The cross-ratio of each* A_i *and* 3 *collinear W's is equianharmonic. Through every W there are* 4 *such lines of collinearity.*

3. *Surfaces on the* G_{27}. It is clear that

$$F = \sum C_{abcdef}(x_1^3 - x_2^3)^a(x_1^3 - x_3^3)^b(x_1^3 - x_4^3)^c(x_2^3 - x_3^3)^d \cdot (x_2^3 - x_4^3)^e(x_3^3 - x_4^3)^f X_{abcdef}^{(g)} = 0,$$

where the X's are quaternary forms of degree g, with $a+b+c+d+e+f=m$ (constant), is a surface of order $3m+g$ with the points of the G_{27} as base points. The simplest form of this kind is $C_1(x_1^3-x_2^3)+C_2(x_1^3-x_3^3)+C_3(x_1^3-x_4^3)+C_4(x_2^3-x_3^3)+C_5(x_2^3-x_4^3)+C_6(x_3^3-x_4^3)=0$. By properly relabeling the coefficients of F, this can be put into the simpler form

$$F_3 = a_1x_1^3 + a_2x_2^3 + a_3x_3^3 + a_4x_4^3 = 0,$$

with $a_1+a_2+a_3+a_4=0$. Among these are four cubic cones with the A_i's as vertices. For example, when $a_4=0$, and $a_1+a_2+a_3=0$, or $a_1=\alpha_2-\alpha_3$, $a_2=\alpha_3-\alpha_1$, $a_3=\alpha_1-\alpha_2$, we have the cubic cone

$$(\alpha_2 - \alpha_3)x_1^3 + (\alpha_3 - \alpha_1)x_2^3 + (\alpha_1 - \alpha_2)x_3^3 = 0,$$

which depends on one effective constant. This agrees with the fact established before that the 27 W's lie by 3 on 9 lines through A_1. The result may be stated as the following theorem.

THEOREM 3. *The cubics on the G_{27} form a linear system of dimension two, or a net. Among these are four cubic cones with the A_i's as vertices. The 9 lines through an A_i, containing the 27 W's, form an associated system of lines of a pencil of cubic cones with the common vertex A_i. The G_{27} is an associated set of base points of the net of cubics.*

Among the quartic F-surfaces may be mentioned in particular

$$F_4 = \sum (a_{ijk}x_k + a_{ijl}x_l)(x_i^3 - x_j^3) = 0,$$

in which i, j, k, l are 1, 2, 3, 4 taken in any order. F_4 depends apparently on 12 homogeneous constants, and so does

$$\begin{aligned} F_4' = {} & (a_2x_2 + a_3x_3 + a_4x_4)x_1^3 + (b_1x_1 + b_3x_3 + b_4x_4)x_2^3 \\ & + (c_1x_1 + c_2x_2 + c_4x_4)x_3^3 + (d_1x_1 + d_2x_2 + d_3x_3)x_4^3 = 0. \end{aligned}$$

But for F_4' to be of the form F_4, it is clear that all the coefficients of F_4' cannot be independent. In fact it is easily seen that the four identities must exist: $b_1+c_1+d_1=0$, $a_2+c_2+d_2=0$, $a_3+b_3+d_3=0$, $a_4+b_4+c_4=0$. From this follows our next theorem.

THEOREM 4. *The quartics F_4 form a linear system of dimension 7, which may be written in the form*

$$\begin{aligned} & \{(\beta_1 - \beta_3)x_2 + (\gamma_1 - \gamma_2)x_3 + (\delta_2 - \delta_3)x_4\}x_1^3 \\ & + \{(\alpha_2 - \alpha_3)x_1 + (\gamma_2 - \gamma_4)x_3 + (\delta_3 - \delta_1)x_4\}x_2^3 \\ & + \{(\alpha_3 - \alpha_4)x_1 + (\beta_3 - \beta_4)x_2 + (\delta_1 - \delta_2)x_4\}x_3^3 \\ & + \{(\alpha_4 - \alpha_2)x_1 + (\beta_4 - \beta_1)x_2 + (\gamma_4 - \gamma_1)x_3\}x_4^3 = 0. \end{aligned}$$

The tangent planes to such an F_4 at the A_i's are concurrent.

Without going into a classification and extended discussion of surfaces on the G_{27} we conclude this section by the construction of the symmetric sextic

$$F_6 = \sum (x_i^3 - x_k^3)^2 = 0, \qquad (i \neq k).$$

Denoting the elementary symmetric functions on the variables by $\phi_1 = \sum x_i$, $\phi_2 = \sum x_i x_k$, $\phi_3 = \sum x_i x_k x_l$, $\phi_4 = x_1x_2x_3x_4$, we find that F_6 has the form

$$\begin{aligned} F_6 = {} & 3\phi_1^6 - 18\phi_1^4\phi_2 + 18\phi_1^3\phi_3 + 27\phi_1^2\phi_2^2 - 24\phi_1^2\phi_4 - 30\phi_1\phi_2\phi_3 \\ & + 24\phi_2\phi_4 - 8\phi_2^3 + 3\phi_3^2 = 0. \end{aligned}$$

Dividing through by ϕ_1^6 and setting $x=\phi_2/\phi_1^2$, $y=\phi_3/\phi_1^3$, $z=\phi_4/\phi_1^4$, we map F_6 upon the cubic monoid

$$24z(x-1)-8x^3-30xy+27x^2+3y^2-18x+18y+3=0,$$

that is, upon a rational surface. As the mapping is birational, we have the following result.

THEOREM 5. *The sextic* $\sum(x_i^3-x_k^3)^2=0$ *is rational and has the points of the* G_{27} *as double points.*

4. *The* G_{36} *of V-Points.* In the plane of the triangle $A_1A_2A_3$ consider the syzygetic pencil of cubics $x_1^3+x_2^3+x_3^3-6\lambda x_1x_2x_3=0$, and the 12 vertices of the four flex triangles, among which are the three vertices $A_1(1,0,0)$, $A_2(0,1,0)$, $A_3(0,0,1)$. When we exclude these, there remain 9 V-points, defined by $V(\omega^\alpha,\omega^\beta,\omega^\gamma,0)$, $\alpha,\beta,\gamma\equiv 0,1,2 \pmod 3$. In a similar way we find 9 V-points in the remaining coordinate planes, hence, altogether 36 V's. A property of these points has already been stated in Theorem 1. There are 27 planes of the type

$$\omega^\alpha x_1+\omega^\beta x_2+\omega^\gamma x_3+x_4=0$$

on each of which we find 8 V's. For example, on the plane $x_1+\omega x_2+\omega^2x_3+x_4=0$ lie

$$(0,1,1,1),\quad(\omega,0,1,1),\quad(\omega,\omega,1,1),\quad(1,1,1,0),$$
$$(0,\omega,\omega^2,1),\quad(\omega^2,0,\omega^2,1),\quad(\omega^2,1,0,1),\quad(1,\omega,\omega^2,0).$$

They also lie on a conic cut out on the plane by the quadric $\omega x_1x_2+\omega^2x_1x_3+x_1x_4+\omega^2x_3x_4+\omega x_2x_4+x_2x_3=0$. A point V, say $(1,\omega,\omega^2,0)$, lies on 6 planes,

$$x_1+x_2+x_3+x_4=0,\qquad x_1+\omega x_2+\omega^2x_3+x_4=0,$$
$$\omega x_1+\omega x_2+\omega x_3+x_4=0,\qquad \omega^2x_1+\omega^2x_2+\omega^2x_3+x_4=0,$$
$$\omega x_1+\omega^2x_2+x_3+x_4=0,\qquad \omega^2x_1+x_2+\omega x_3+x_4=0.$$

THEOREM 6. *The* 36 *V's lie by* 8 *on* 27 *conics in* 27 *planes. Through each V pass* 6 *of these planes and one of the* 36 *lines through the* A_i*'s containing each* 3 *of the W's.*

As before we may again set up linear systems of surfaces on the G_{36} (including the A_i's), and eventually on the G_{36} and the G_{27}. We shall restrict ourselves to one particularly interesting example of a septimic which cuts the faces of the coordinate tetrahedron outside its edges each in a *Caporali quartic, and contains both the* G_{27} *and the* G_{36}.

$$\begin{aligned} F_7 = {} & x_1x_2x_3\{a_{14}x_1(x_2^3 - x_3^3) + a_{24}x_2(x_3^3 - x_1^3) + a_{34}x_3(x_1^3 - x_2^3)\} \\ & + x_1x_2x_4\{a_{13}x_1(x_2^3 - x_4^3) + a_{23}x_2(x_4^3 - x_1^3) + a_{43}x_4(x_1^3 - x_2^3)\} \\ & + x_1x_3x_4\{a_{12}x_1(x_3^3 - x_4^3) + a_{32}x_3(x_4^3 - x_1^3) + a_{42}x_4(x_1^3 - x_3^3)\} \\ & + x_2x_3x_4\{a_{21}x_2(x_3^3 - x_4^3) + a_{31}x_3(x_4^3 - x_2^3) + a_{41}x_4(x_2^3 - x_3^3)\} = 0. \end{aligned}$$

It has the A_i*'s as triple points and the* $\overline{A_iA_k}$ *as single lines.*

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EINSTEIN SPACES OF CLASS ONE*

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1. *Introduction.* An Einstein space is defined as a Riemann space for which

$$R_{\alpha\beta} = \frac{R}{n} g_{\alpha\beta}. \tag{1}$$

We assume the first fundamental form

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta \tag{2}$$

to be non-singular, but do not restrict ourselves to the positive definite case. An $n+1$ dimensional space is said to be flat when its first fundamental form can be reduced to†

$$ds^2 = \sum_{i=1}^{n+1} c_i(dx^i)^2, \tag{3}$$

where the c_i are definitely plus one or minus one. An n dimensional Riemann space which is not flat is said to be of class one if it can be imbedded in an $n+1$ dimensional flat space. The purpose of this paper is to determine necessary and sufficient conditions that an Einstein space be of class one.

There is no problem when $n=2$, for then every space which

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† Throughout this paper Latin indices will have the range 1 to $n+1$; Greek indices the range 1 to n.