AN APPLICATION OF DERIVATIVES OF NON-ANALYTIC FUNCTIONS IN PLANE STRESS PROBLEMS*

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1. *The Plane Stress Problem.* A plane state of stress is defined in a region in the xy plane by an Airy's stress function $F(x, y)$, where F satisfies

$$\nabla^4 F(x, y) = \nabla^2\nabla^2 F = \nabla^2(F_{xx} + F_{yy}) = 0. \tag{1}$$

The normal stresses σ_x and σ_y in the directions of x and y, respectively, and the corresponding shearing stress τ_{xy} are obtained from F when no body forces are present by the following:†

$$\sigma_x = F_{yy}, \qquad \sigma_y = F_{xx}, \qquad \tau_{xy} = -F_{xy}. \tag{2}$$

Equilibrium conditions show that the stress tensor at any point may be referred to any set of orthogonal planes by the relations

$$\begin{aligned} \sigma_{x'}, \sigma_{y'} &= \frac{\sigma_x + \sigma_y}{2} \pm \frac{\sigma_x - \sigma_y}{2} \cos 2\theta \pm \tau_{xy} \sin 2\theta, \\ \tau_{x'y'} &= \frac{\sigma_y - \sigma_x}{2} \sin 2\theta + \tau_{xy} \cos 2\theta, \end{aligned} \tag{3}$$

where the $x'y'$ axes have been rotated through the positive angle θ from the xy axes.

2. *The Stress Circle and Kasner's Derivative Circle.* A graphical construction due to Mohr‡ is frequently employed in place of equations (3). In the complex plane $\gamma = \sigma + i\tau$, describe a circle having its center on the σ axis and passing through the points (σ_x, τ_{xy}) and $(\sigma_y, -\tau_{xy})$ which are designated respectively as points C and E. Then, corresponding to a counter-clockwise ro-

* Presented to the Society, November 28, 1936.

† Subscripts on F denote partial derivatives with respect to the indicated variables. The subscripts on the stresses σ and τ refer to directions along which the stresses act.

‡ O. Mohr, *Abhandlungen aus dem Gebiete der Technische Mechanik*, 2d edition, 1914.

tation θ of the x and y axes, the diameter CE rotates clockwise through an angle 2θ into a new position $C'E'$. The points C' and E' then have the new coordinates $(\sigma_{x'}, \tau_{x'y'})$ and $(\sigma_{y'}, -\tau_{x'y'})$ defined in (3).

For every point in a stressed region there exists a Mohr's dyadic stress circle which describes the complete state of stress at that point. One observes that as θ varies in the xy plane, a characteristic point E in the γ plane moves around the circle at twice the rate at which θ changes and in the opposite sense.

It may be recalled that the property ascribed to E is identical with that given by Kasner* for γ, where γ is defined as the directional derivative of a non-analytic function of a complex variable. The circle described by γ has been called by Hedrick† the Kasner circle. Through the similarity of properties of the Kasner circle for the directional derivative and of Mohr's circle for a state of stress, one is led to seek the function of a complex variable which will lead through its directional derivatives to a family of Mohr's circles.

3. *Properties of the $H(z, \bar{z})$ Function.* Let the Airy's function $F(x, y)$ be written

$$F(x, y) = F\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) = F(z, \bar{z}). \tag{4}$$

Equations (1) and (2) become

$$F_{zz\bar{z}\bar{z}} = 0, \tag{5}$$

$$\sigma_x, \sigma_y = 2F_{z\bar{z}} \mp (F_{zz} + F_{\bar{z}\bar{z}}), \qquad \tau_{xy} = -i(F_{zz} - F_{\bar{z}\bar{z}}). \tag{6}$$

Let the function H and its directional derivative be defined by

$$H(z, \bar{z}) = 2F_{\bar{z}}, \tag{7}$$

$$\gamma_H = H_z + H_{\bar{z}}e^{-2i\theta} = \frac{\sigma_x + \sigma_y}{2} - \left(\frac{\sigma_x - \sigma_y}{2} + i\tau_{xy}\right)e^{-2i\theta}. \tag{8}$$

It is evident from (8) that γ_H represents the point E' on Mohr's circle and that the congruence of Kasner circles for the

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* Kasner, Science, vol. 66 (1927), pp. 581–582.

† E. R. Hedrick, *Non-analytic functions of a complex variable*, this Bulletin, vol. 39 (1933).

function H is identical with the family of Mohr's circles for the stresses defined by F. Furthermore, the *principal directions* and *characteristic curves*, as defined for the non-analytic function H, coincide respectively with the directions of the principal stresses and the stress trajectories in the stress field of F.

The significance of γ_H, determined graphically from (8), may also be determined algebraically by solving (3) for σ_x, σ_y, and τ_{xy} in terms of $\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$. Substituting these results in (8), one obtains the relations

$$(9)\qquad \gamma_H = \sigma_{y'} - i\tau_{x'y'}, \qquad i\gamma_H = \gamma_{iH} = \tau_{x'y'} + i\sigma_{y'}.$$

The following properties are immediately deducible:

A. *The directional derivative of iH, taken tangent to any arc, is the resultant unit stress acting upon that arc.*

B. *The directional derivative of H, taken tangent to a stress trajectory, is real; conversely, if it is real the arc is acted upon by normal stresses only.*

C. *The directional derivative of H, taken tangent to a stress-free boundary, vanishes everywhere along that boundary.*

These properties are useful in discussing the nature of stresses on curvilinear boundaries. Furthermore, there is a physical reason for taking the directional derivatives of second and higher orders in such a manner that the slope of the curve of approach remains constant at any point while the order of the derivative increases. As pointed out by Hedrick,* this is only one of the possible choices of a definition for derivatives of second and higher orders.

4. *Application to Conformal Mapping.* Let a given region in the complex plane $w=u+iv$, having stresses defined by an Airy's function $\mathfrak{F}(w, \bar{w})$, be mapped conformally into a region in the z plane by

$$(10)\quad z = g(w), \quad \text{with } h = \left|\frac{dz}{dw}\right| = (g'\bar{g}')^{1/2}, \quad \text{and } g' = \frac{dg}{dw},$$

and where $\bar{g}'$ is formed from g' by replacing i by $-i$. At a given point $w=w_1$ on the boundary of the stressed region in the w plane, let the tangent to the boundary make an angle θ_1 with

* Loc. cit.

the line $v=$constant drawn through w_1. When the point w_1 maps into z_1, the tangent to the new boundary at z_1 makes an angle $\theta=\theta_1+\theta_2$ with the line $y=$constant. Since the mapping is conformal, θ_2 is the angle between the line $y=$constant and the tangent at z_1 to the mapped arc of $v=$constant. Then

$$g' = he^{i\theta_2}. \tag{11}$$

If the original boundary is free from stress, then condition C applies in the w plane and may be written

$$\gamma_{\mathfrak{H}}\Big]_{\theta=\theta_1} = \mathfrak{F}_{w\bar{w}} + \mathfrak{F}_{\bar{w}\bar{w}}e^{-2i\theta_1} = 0. \tag{12}$$

Since both the real and imaginary parts of (12) must be zero, the difference of the real and imaginary parts becomes

$$\mathfrak{F}_{w\bar{w}} + \mathfrak{F}_{ww}e^{2i\theta_1} = 0. \tag{13}$$

Multiplication of (12) and (13) by $e^{-2i\theta_1}$ yields two similar restrictions on $\mathfrak{F}$ at the stress-free boundary.

As an illustration of the use of the foregoing, consider Michell's* problem of the inversion of a field of stress. The mapping function is

$$zw = 1, \tag{14}$$

with the additional relation that the new stress function, $F(z, \bar{z})$, applicable to the new region in the z plane is given by

$$F(z, \bar{z}) = z\bar{z}\mathfrak{F}\left(\frac{1}{z}, \frac{1}{\bar{z}}\right) = z\bar{z}\mathfrak{F}(w, \bar{w}). \tag{15}$$

By means of the non-analytic functions

$$H = 2F_{\bar{z}}, \qquad \text{and} \qquad \mathfrak{H} = 2\mathfrak{F}_{\bar{w}},$$

the results of Michell may be obtained, namely, a stress-free boundary maps into a boundary acted upon by a constant normal stress, and stress trajectories map into stress trajectories.

The directional derivative of H, taken along a boundary in the z plane, is

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* J. H. Michell, *The inversion of plane stress*, Proceedings of the London Mathematical Society, vol. 34 (1901).

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\begin{aligned}
\gamma_H \Big]_{\theta=\theta_1+\theta_2} &= 2\left[F_{z\bar{z}} + F_{\bar{z}\bar{z}}e^{-2i(\theta_1+\theta_2)}\right] \\
&= \frac{2}{h}\left[-h\mathfrak{F} + \frac{1}{\bar{w}}\mathfrak{F}_w + \frac{1}{w}\mathfrak{F}_{\bar{w}} - (\mathfrak{F}_{w\bar{w}} + \mathfrak{F}_{\bar{w}\bar{w}}e^{-2i\theta_1})\right] \qquad (16) \\
&= 2(-\mathfrak{F} + w\mathfrak{F}_w + \bar{w}\mathfrak{F}_{\bar{w}}) - \frac{2}{h}\left[\gamma_{\mathfrak{H}}\right]_{\theta=\theta_1}.
\end{aligned}
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This equation is obtained by using (14), (15) and the operators

$$h^2 F_{z\bar{z}} = F_{w\bar{w}}, \qquad (\bar{g}')^3 F_{\bar{z}\bar{z}} = \bar{g}' F_{\bar{w}\bar{w}} - \bar{g}'' F_{\bar{w}}.$$

From the boundary condition (12), equation (16) yields

$$\gamma_H \Big]_{\theta=\theta_1+\theta_2} = 2(-\mathfrak{F} + w\mathfrak{F}_w + \bar{w}\mathfrak{F}_{\bar{w}}) = \text{a real function}, \qquad (17)$$

since $\mathfrak{F}$ is real. Therefore, from the condition B, the boundary is acted upon by normal stresses only.

To determine the nature of the normal stresses on the new boundary, one finds the directional derivative of (17) along the boundary to be

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\begin{aligned}
\frac{d\gamma_H}{dz}\Big]_{\theta=\theta_1+\theta_2} &= -2w^3(\mathfrak{F}_{ww} + \mathfrak{F}_{w\bar{w}}e^{-2i\theta_1}) \qquad (18) \\
&\quad - 2\bar{w}w^2(\mathfrak{F}_{w\bar{w}} + \mathfrak{F}_{\bar{w}\bar{w}}e^{-2i\theta_1}) = 0.
\end{aligned}
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Equation (18) is obtained from (17) by transforming from the independent variables z and $\bar{z}$ to w and $\bar{w}$. The boundary conditions similar to (12) and (13) show that the directional derivative vanishes on the boundary, and thus the normal stresses which act upon the transformed boundary must be constant.

In the w plane let θ_1 now designate the angle between the u axis and the tangent to a stress trajectory at any point w_1 within the stressed region. Then, by condition B, the directional derivative of $\mathfrak{H}$ along the stress trajectory, $\gamma_{\mathfrak{H}}]_{\theta=\theta_1}$ is real; hence, by (16), at any point z_1 on the mapped trajectory, the directional derivative, $\gamma_H]_{\theta=\theta_1+\theta_2}$, is also real. Therefore, by the second part of condition B, the original stress trajectories map into stress trajectories for the new stress field.

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