CURVES BELONGING TO PENCILS OF LINEAR LINE COMPLEXES IN S_4

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1. *Introduction.* It has been demonstrated in at least two ways* that every curve in S_3, whose tangents belong to a non-special linear line complex can be mapped into a curve in S_3 all of whose tangents meet a fixed conic. In this paper, similar theorems are obtained for curves in S_4 whose tangents belong to (1) a single linear complex, (2) a pencil of linear complexes.

In what follows we shall use the symbol Γ to represent a non-special complex, that is, a complex which does not consist of the totality of lines which meet a plane. We shall use the symbol Π to represent a pencil of complexes which does not contain any special complexes. The customary symbol V_m^r will be used to represent a variety of order r and of dimension m.

2. *Hyperpencil of Lines.* We note first that no curve lying in S_4 but in no linear subspace of S_4 can belong to a special complex. For all the tangents of such a curve would have to meet the singular plane of the complex, which would require the osculating S_3's of the curve to contain the plane. This is impossible unless the curve lies entirely in an S_3 containing the singular plane. We are thus concerned with non-special complexes in (1) and with pencils which contain no special complexes in (2).

Through an arbitrary point of S_4 pass ∞^2 lines belonging to a non-special complex Γ. These lines lie in an S_3, the polar S_3 of the point as to Γ, and form what we shall call a hyperpencil of lines. For every complex Γ, there is a unique point with the property that every line which passes through that point belongs to Γ. We shall call this point the vertex of Γ. Of the five types of pencils of complexes in S_4 all but one contain special complexes. The one admissible type, Π, consists of ∞^1 complexes whose vertices lie on a non-composite conic, K. Through an ar-

* V. Snyder, *Twisted curves whose tangents belong to a linear complex*, American Journal of Mathematics, vol. 29 (1907), pp. 279–288.

C. R. Wylie, Jr., *Space curves belonging to a non-special linear line complex*, American Journal of Mathematics, vol. 57 (1935), pp. 937–942.

bitrary point of S_4 pass ∞^1 lines of Π, forming a plane pencil. Through a point of K pass ∞^2 lines of Π. These lines lie in an S_3, and thus form a hyperpencil. Only one line of an arbitrary plane field belongs to Π, while all lines in the plane, σ, of K belong to Π.

3. *The Associated V_6^5 in S_9.* If the ten Grassman coordinates of the lines of S_4 be regarded as point coordinates in S_9, the five quadratic identities which exist among the line coordinates define a variety which is known to be of order five and of dimension six. The lines of S_4 are represented* in S_9 by the points of this V_6^5. A ruled surface in S_4 is represented in S_9 by a curve on V_6^5. If the ruled surface is developable, not only the image curve but its tangent developable lies on V_6^5. The tangents to the image curve in this case are the images of the pencils of lines lying in the osculating planes of the cuspidal edge of the developable in S_4, and having their vertices at the points of osculation. A linear complex is represented in S_9 by the V_5^5 common to V_6^5 and the S_8 which the equation of the complex defines. If a curve in S_4 belongs to a linear complex its image curve, that is, the image curve of its tangent developable, lies with its tangents on the V_5^5 which represents the complex.

Through the vertex of a complex, Γ, pass ∞^3 planes each of which contains ∞^2 lines of Γ. On V_5^5 these are represented by planes. Suppose there is on V_5^5 a curve C' and its tangent developable, the image of a curve C in S_4 which belongs to Γ. Working now in the S_8 given by the equation of the complex, Γ, let us project this configuration from one of the planes, ω', of V_5^5 upon an S_5. The singular elements in the projection are the ∞^2 planes which meet ω' in a line. These are the images of the hyperpencils of lines belonging to Γ which issue from the points of ω, the plane field of lines in S_4 whose image in S_9 is ω'. Each tangent to C' meets one of these singular planes, because in S_4 each osculating plane of C meets ω in a point, and hence there is in each osculating plane one line which passes through the point of osculation and belongs to a hyperpencil whose vertex is in ω. The configuration of C' and its tangents will thus project into a curve C'' in S_5 all of whose tangents meet the surface which is the projection of the singular planes.

To determine the order of this surface consider the polar

* Compare W. L. Edge, *Ruled Surfaces*, 1931, §2.

S_3's as to Γ of the points of ω. There are only ∞^1 of these S_3's, for all points of ω collinear with the vertex of the complex have the same polar. These S_3's set up a $1:1$ correspondence between the points of an arbitrary line L, and the lines in ω which pass through the vertex of Γ. Moreover, the line joining any point P of ω to the point of L corresponding to the line through P and the vertex of Γ determines with the pencil of lines of ω which pass through P a hyperpencil whose image is a singular plane of the projection. When sectioned by a general linear complex, the double infinity of lines which join the points of ω to their corresponding points on L yields the single infinity of lines joining corresponding points of L and a conic in ω. Such a family of lines is evidently of order three, hence the lines which determine with the pencils of lines of ω the hyperpencils whose images are singular planes of the projection are represented on V_5^5 by a cubic surface. This projects into a cubic surface in S_5; hence we have the following theorem.

THEOREM 1. *Every curve in S_4 whose tangents belong to a non-special linear line complex can be mapped into a curve in S_5 all of whose tangents meet a cubic surface.*

We have already noted that the five quadric hypersurfaces which are defined by the five quadratic identities existing among the coordinates of the lines of S_4 intersect in a V_6^5 and not in a V_4^{32} as would be the case in general. Since the V_5^5 which is the image of Γ is obtained from V_6^5 by sectioning the latter with an S_8, it follows that V_5^5 is determined by five V_7^2's. From this fact it is evident that the projection can be reversed, and that any curve in S_5 whose tangents meet the cubic surface which is the projection of the singular planes can be mapped into a curve in S_4 which belongs to a linear complex.

4. *Map of a Curve in S_4.* If a curve C of S_4 belongs to an admissible pencil of complexes, Π, its image curve, C', lies with its tangents on the V_4^5 which is the image of Π, and which is defined by V_6^5 and the S_7 given by the equations of Π. Let us project such a configuration upon an S_4 from the plane σ' which is the image of the lines of the plane σ of K, the locus of vertices of the complexes of Π. The singular elements in the projection are the planes of V_4^5 which meet σ' in a line, namely, the planes

which are the images of the ∞^1 hyperpencils of lines belonging to Π which issue from the points of K.

Now the lines which lie in the osculating planes of C and pass through the points of osculation all belong to Π; likewise all lines of σ belong to Π. Hence at every point where an osculating plane of C meets σ there are three non-coplanar lines of Π, and hence ∞^2 lines of Π. But the only points of σ through which pass ∞^2 lines of Π are the points of K. Since every osculating plane of C meets σ, we have the following theorem.

THEOREM 2. *The osculating planes of every curve of S_4 belonging to a pencil of linear line complexes which contains no special complexes, meet a fixed conic.*

The converse of this theorem is not true, as the following example shows. The osculating planes of the curve

$$x_1 : x_2 : x_3 : x_4 : x_5 = 45t^4 : 18t^5 : 10t^6 : -20t^3 : 1$$

meet the conic $x_2^2 = x_1x_3$, $x_4 = 0$, $x_5 = 0$, but the curve belongs to but one complex.

From this theorem it follows that every tangent to C' meets in a point one of the planes which are singular in the projection. The projection of these planes is a curve whose order can be found by considering the 1:1 correspondence set up between the points P of the conic K and the points P' of an arbitrary line L, by the polar S_3's as to Π of the points P. Each hyperpencil whose image is a singular plane of the projection is determined by the lines of σ which pass through one of the points of K, together with the line joining this point of K to its corresponding point on L. The lines joining corresponding points of K and L form a cubic regulus whose image on V_4 is a cubic curve. This projects into a cubic curve in S_4; hence we have the following theorem.

THEOREM 3. *Every curve in S_4 whose tangents belong to a pencil of linear line complexes containing no special complexes can be mapped into a curve in S_4 all of whose tangents meet a fixed cubic curve.*

Evidently this process can be reversed, and a curve in S_4 whose tangents meet a fixed cubic can be mapped into a curve in S_4 belonging to a pencil of linear complexes.

5. *Equations of a Curve in* Γ *or* Π. If the equation of Γ be taken as $P_{13}+P_{24}=0$, the equation of a curve belonging to Γ can be written down at once from the results* for three dimensions:

$$\text{A:} \quad x_1 = t, \quad x_2 = tf' - 2f, \quad x_3 = f', \quad x_4 = 1, \quad x_5 = g,$$

where f and g are arbitrary functions of t, and the primes indicate differentiation with respect to t. If Π be chosen as $P_{13}+P_{24}=0$, $P_{12}+P_{45}=0$, the equations of C are found to be

$$\text{B:} \quad \begin{aligned} x_1 &= t, & x_2 &= tF'' - 2F', & x_3 &= F'', \\ x_4 &= 1, & x_5 &= -t^2F'' + 4tF' - 6F, \end{aligned}$$

where $F=\int f(t)dt$, $f(t)$ has the same significance it had in equations A, and the primes indicate differentiation with respect to t.

6. *Bundles of Complexes in* S_4. Of the fifteen types of bundles of complexes in S_4† all but one contain special complexes. A bundle of the admissible type consists of ∞^2 complexes, the locus of whose vertices is a quartic surface in S_4. The lines belonging to such a bundle are all trisecants of the locus of vertices. Of the triple infinity of these trisecants, a double infinity are tangents, and a single infinity are inflexional tangents. Through an arbitrary point of S_4 passes a unique line of the bundle. Through each point of the locus of vertices pass ∞^1 lines of the bundle, forming a plane pencil. Thus those curves, if any, whose tangents belong to the bundle must lie on the locus of vertices. Segre‡ has shown that there is a unique curve, the rational normal quartic in fact, belonging to a bundle of this type. This quartic curve is just the locus of points of contact of the ∞^1 inflexional tangents of the locus of vertices.

Systems of complexes of more than two degrees of freedom cannot contain curves of S_4.

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* C. R. Wylie, Jr., loc. cit.; B. Segre, *Sulle curve le cui tangenti appartengono al massimo numero di complessi lineari independenti*, Memorie dell'Accademia dei Lincei, (6), vol. 2 (1928), pp. 578–592.

† R. Weitzenbock, *Zum System von drei Strahlenkomplexen im vier dimensionalen Raum*, Monatshefte für Mathematik und Physik, vol. 21 (1910), pp. 103–124.

‡ B. Segre, loc. cit.