ON THE SUMMABILITY OF FOURIER SERIES

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1. *Introduction.* It is well known that the Abel method of summability is stronger than the Cesàro methods of any order. An example has been given* to show that there are series which are Abel summable but not Cesàro summable for any order. This series is one for which $a_n \neq o(n^\alpha)$ for any α, and hence which cannot be (C, α) summable for any α. This series cannot be a Fourier series since for all Fourier series $a_n = o(1)$. We propose to give an example of the existence of a Fourier series which is Abel summable but not Cesàro summable.

We shall make use of some results of Paley† which show that, if the Fourier series of $f(x)$,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{1}$$

is (C, α) summable at the point x, then, for $\beta > \alpha$,

$$\begin{aligned} R_\beta(f, t) &= \beta \int_0^t \{f(x + \tau) + f(x - \tau) - 2f(x)\}(t - \tau)^{\beta-1} d\tau \\ &= o(t^\beta), \qquad \text{as} \qquad t \to 0, \end{aligned}$$

and conversely, if $R_\alpha(f, t) = o(t^\alpha)$, as $t \to 0$, then the series (1) is (C, β) summable for every $\beta > \alpha + 1$. We shall first show that for every $n > 1$ there is a function $f_n(x)$ such that at $x = 0$

$$\overline{\lim_{t\to 0}} \left| \frac{1}{t^j} R_j(f_n, t) \right| = \infty, \qquad (j \leqq n - 1), \tag{2}$$

but

$$R_n(f_n, t) = o(t^n), \qquad \text{as} \qquad t \to 0. \tag{3}$$

This implies that the Fourier series of $f_n(x)$ is $(C, n+2)$ summable at $x = 0$ and therefore Abel summable. The function

* See Landau, *Darstellung und Begründung einiger neuer Ergebnisse der Funktionentheorie*, 1929, p. 51.

† R. E. A. C. Paley, *On the Cesàro summability of Fourier series and allied series*, Proceedings of the Cambridge Philosophical Society, vol. 26 (1929), pp. 173–203.

$$f(x) = \sum_{n=2}^{\infty} d_n f_n(x)$$

is then defined with the d_n's so chosen that the Fourier series of $f(x)$ is Abel summable, but for every n

$$R_n(f, t) \neq o(t^n), \qquad \text{as} \qquad t \to 0.$$

This implies, by the theorem of Paley, that the Fourier series of $f(x)$ cannot be (C, α) summable for any α.

2. *Properties of $f_n(x)$.* We suppose for the moment that n is fixed and we let $c = (1 + 1/(n - 1/2))$. We define $a_\nu = 2^{-c\nu}$, $b_\nu = 2^{-\nu} - a_\nu$; then, if $\nu \geqq n$, $b_\nu > 2^{-(\nu+1)}$, so that the intervals $(b_\nu, 2^{-\nu})$ are non-overlapping for $\nu \geqq n$. We define

$$f_n(x) = \begin{cases} 2^\nu, \quad b_\nu \leqq |x| \leqq b_\nu + \dfrac{a_\nu}{2^n}, & (\nu = n, n+1, \cdots), \\ -f_n\left(x - 2^j \dfrac{a_\nu}{2^n}\right), \quad b_\nu + 2^j \dfrac{a_\nu}{2^n} < |x| \leqq b_\nu + 2^{j+1} \dfrac{a_\nu}{2^n}, & \\ \qquad (j = 0, \cdots, n-1; \nu = n, \cdots), & \\ 0, \quad \text{elsewhere on } (-\pi, \pi). & \end{cases}$$

Then $f_n(x) \subset L$ on $(-\pi, \pi)$, for

$$\int_{-\pi}^{\pi} |f_n(x)|\, dx = 2 \sum_{\nu=n}^{\infty} 2^\nu a_\nu = 2 \sum_{\nu=n}^{\infty} 2^{-\nu/(n-1/2)} < \infty.$$

At $x = 0$, $f_n(x+t) + f_n(x-t) - 2f_n(x) = 2f_n(t)$. We have

$$\int_{b_\nu}^{b_\nu + 2(a_\nu/2^n)} f_n(t)dt = \int_{b_\nu}^{b_\nu + a_\nu/2^n} f_n(t)dt - \int_{b_\nu}^{b_\nu + a_\nu/2^n} f_n(t)dt = 0.$$

By the definition of $f_n(x)$,

$$f_n(t) = -f\left(t - 2^j \cdot \frac{a_\nu}{2^n}\right), \quad b_\nu + 2^j \frac{a_\nu}{2^n} < t \leqq b_\nu + 2^j \frac{a_\nu}{2^n},$$

so that by induction

$$\int_{b_\nu}^{b_\nu + 2^j(a_\nu/2^n)} f_n(t)dt = 0, \qquad (1 \leqq j \leqq n);$$

and therefore, if $b_\nu + 2^j(a_\nu/2^n) < t$, $(1 \leqq j \leqq n-1)$,

$$R_1(f_n, t) = 2\int_{b_\nu+2^j(a_\nu/2^n)}^{t} f_n(\tau)d\tau.$$

Hence, if $b_\nu + 2^j 2(a_\nu/2^n) < t < b_\nu + 2^j 2(a_\nu/2^n)$, $(0 \leqq j \leqq n-2)$,

$$R_1(f_n, t) = -R_1\left(f_n, t - 2^j \cdot 2 \cdot \frac{a_\nu}{2^n}\right).$$

Since

$$R_{k+1}(f_n, t) = (k+1)\int_0^t R_k(f_n, \tau)d\tau,$$

we see that in the same way, if $t > b_\nu$,

$$\frac{1}{k+1} R_{k+1}(f_n, t) = \int_{b_\nu}^t R_k(f_n, \tau)d\tau,$$

and, for

$$b_\nu + 2^j \cdot 2^{k+1}(a_\nu/2^n) < t < b_\nu + 2^{j+1}\cdot 2^{k+1}(a_\nu/2^n),\ (j+k \leqq n-2),$$

$$R_{k+1}(f_n, t) = -R_{k+1}\left(f_n, t - 2^j \cdot 2^{k+1} \cdot \frac{a_\nu}{2^n}\right).$$

Therefore, for $k \leqq n-1$,

$$R_k\left(f_n, b_\nu + \frac{a_\nu}{2^n}\right) = 2k2^\nu \int_0^{a_\nu/2^n} \left(\frac{a_\nu}{2^n} - t\right)^{k-1} dt$$

$$= 2^{\nu+1}\left(\frac{a_\nu}{2^n}\right)^k \neq o(2^{\nu k}) \qquad \text{as} \qquad \nu \to \infty.$$

Finally, if $b_\nu \leqq t < 2^{-\nu}$,

$$R_n(f_n, t) = 2n\int_0^t f_n(\tau)(t-\tau)^{n-1}d\tau = O\left(2^\nu \int_{b_\nu}^t (t-\tau)^{n-1}d\tau\right)$$

$$= O(2^\nu a_\nu{}^n) = O(2^\nu 2^{-n\nu/(n-1/2)} 2^{-n\nu}) = o(2^{-n\nu}) = o(t) \text{ as } t \to 0.$$

Therefore the function $f_n(x)$ has the properties (2) and (3).

3. *A Function whose Fourier Series is not Summable* (C, α). As we have already mentioned, the Fourier series of $f_n(x)$ will be Abel summable at $x = 0$. Therefore,

$$A_n = \underset{0 \leq r<1}{\text{l.u.b.}}\ A(f_n, r)$$
$$= \underset{0 \leq r<1}{\text{l.u.b.}}\ \frac{1}{2\pi}\int_0^{\pi} \{f_n(x+t)+f_n(x-t)-2f_n(x)\}\ \frac{1-r^2}{1-2\cos t+r^2}\, dt$$

will exist. We may define two sequences $\{d_n\}$ and $\{t_n\}$ simultaneously by induction so that

$$(4) \qquad d_n \leqq \min\left(\frac{1}{2^n A_n}, \frac{1}{2^n}, \frac{1}{2^n \int_{-\pi}^{\pi} | f_n(t) | \, dt}\right),$$

$$(5) \qquad d_n \leqq \frac{1}{2^n} \min_{\nu \leqq n-2}\left(\frac{1}{t_{\nu+1}^{-\nu} R_\nu(f_n, t_{\nu+1})}\right),$$

$$(6) \qquad | t_n^{-(n-1)} R_{n-1}(f_n, t_n) | > \frac{n}{d_n},$$

$$(7) \qquad | t_n^{-(n-1)} R_{n-1}(f_n, t_n) | < \frac{1}{n}, \qquad (\nu \leqq n-1).$$

It is clear that d_n can be chosen so as to satisfy (4) and (5). It is possible to choose t_n satisfying (6) and (7), since

$$\overline{\lim_{t\to 0}}\ | t^{-(n-1)} R_{n-1}(f_n, t) | = \infty,$$

and

$$t^{-\mu} R_\mu(f_n, t) = o(1) \quad \text{as} \quad t \to 0, \qquad \text{for} \qquad \mu \geqq n.$$

The function

$$f(x) = \sum_{n=2}^{\infty} d_n f_n(x)$$

is integrable, for

$$\int_{-\pi}^{\pi} | f(x) | \, dx \leqq \sum_{n=2}^{\infty} d_n \int_{-\pi}^{\pi} | f_n(x) | \, dx \leqq \sum_{n=2}^{\infty} 2^{-n}.$$

The Fourier series of $f(x)$ is Abel summable, since

$$A(f, r) = \sum_{n=2}^{\infty} d_n A(f_n, r),$$

and $d_n A(f_n, r) \leqq 1/2^n$, and $A(f_n, r) \to 0$ as $r \to 1$, which implies that $A(f, r) \to 0$ as $r \to 1$.

We shall show that, for every n, $R_n(f, t) \neq o(t^n)$, as $t \to 0$. Let us suppose that, for some n, $R_n(f, t) = o(t^n)$, as $t \to 0$; then, since

$$R_{n+1}(f, t) = (n + 1) \int_0^t R_n(f, \tau) d\tau,$$

there would be a constant K such that for all t and $m \geqq n$ we would have

$$| R_m(f, t) | \leqq K t^m. \tag{8}$$

We shall show that for every n

$$| t_n^{-(n-1)} R_{n-1}(f, t_n) | > n + o(1), \qquad \text{as} \qquad n \to \infty,$$

which contradicts (8). We have

$$\begin{aligned} t_n^{-(n-1)} R_{n-1}(f, t_n) &= \sum_{\nu=2}^{\infty} d_\nu t_n^{-(n-1)} R_{n-1}(f_\nu, t_n) \\ &= \sum_{\nu=2}^{n-1} d_\nu t_n^{-(n-1)} R_{n-1}(f_\nu, t_n) + d_n t_n^{-(n-1)} R_{n-1}(f_n, t_n) \\ &\quad + \sum_{\nu=n+1}^{\infty} d_\nu t_n^{-(n-1)} R_{n-1}(f_\nu, t_n). \end{aligned}$$

By (7),

$$\left| \sum_{\nu=2}^{n-1} d_\nu t_n^{-(n-1)} R_{n-1}(f_\nu, t_n) \right| < \frac{1}{n} \sum_{\nu=2}^{n-1} | d_\nu | = o(1), \quad \text{as} \quad n \to \infty,$$

and, by (5),

$$\left| \sum_{\nu=n+1}^{\infty} d_\nu t_n^{-(n-1)} R_{n-1}(f_\nu, t_n) \right| \leqq \sum_{\nu=n+1}^{\infty} 2^{-n} = o(1), \quad \text{as} \quad n \to \infty,$$

so that, by (6),

$$\begin{aligned} | t_n^{-(n-1)} R_{n-1}(f, t_n) | &= | d_n t_n^{-(n-1)} R_{n-1}(f_n, t_n) | + o(1) \\ &> n + o(1), \quad \text{as} \quad n \to \infty. \end{aligned}$$

Therefore by the theorem of Paley the Fourier series of $f(x)$ cannot be (C, α) summable for any α.

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