A MEAN VALUE THEOREM FOR POLYNOMIALS AND HARMONIC POLYNOMIALS

BY J. L. WALSH

1. *Introduction.* We define a polynomial in z of degree n as any function that can be expressed in the form $a_0z^n+a_1z^{n-1}+\cdots+a_n$; we do not require $a_0\neq 0$. With this definition the following theorems are valid, as is our purpose to show in the present note:

THEOREM 1. *If $f(z)$ is an analytic function of z for the value $z=z_0$, then we have*

$$\lim_{h\to 0}\frac{f(z_0+h)+f(z_0+\omega h)+\cdots+f(z_0+\omega^{N-1}h)-Nf(z_0)}{h^N} = \frac{f^{(N)}(z_0)}{(N-1)!}, \tag{1}$$

where ω denotes the number $e^{2\pi i/N}$.

A function $f(z)$ is said to have the *polygonal mean value property* or more simply *the mean value property* if for fixed N and for every z_0 the value of $f(z_0)$ is the mean of the values of $f(z)$ at the vertices of every regular polygon of N sides whose center is z_0.

THEOREM 2. *A necessary and sufficient condition that a function analytic for all values of z have the mean value property is that it be a polynomial of degree $N-1$.*

THEOREM 3. *A necessary and sufficient condition that a real function $f(z)\equiv u(x, y)$ continuous for all values of z $(=x+iy)$ have the mean value property is that it be a harmonic polynomial of degree $N-1$.*

A harmonic polynomial in x and y of degree $N-1$ is defined as the real part of a polynomial in z of degree $N-1$.

2. *Proof of Theorem* 1. In preparation for the proof of Theorem 1 we first formulate a following well known and easily proved lemma.

LEMMA. *If ω denotes the number $e^{2\pi i/N}$, the sum $1+\omega^m+\omega^{2m}+\cdots+\omega^{(N-1)m}$ has the value N or 0 according as m is or is not a multiple of N.*

The result is obvious if m is a multiple of N; in the contrary case we need merely write

$$1+\omega^m+\omega^{2m}+\cdots+\omega^{(N-1)m}=\frac{1-\omega^{Nm}}{1-\omega^m}=0.$$

Let the function $f(z)$ be analytic at the point z_0, hence analytic at every point interior to some circle whose center is z_0. At every such point the Taylor development

$$f(z_0+h)\equiv f(z_0)+\frac{f'(z_0)}{1!}h+\frac{f''(z_0)}{2!}h^2+\cdots$$

is valid, hence by the lemma the relation

$$\begin{aligned} &f(z_0+h)+f(z_0+\omega h)+\cdots+f(z_0+\omega^{N-1})h-Nf(z_0)\\ &\equiv N\left[\frac{f^{(N)}(z_0)}{N!}h^N+\frac{f^{(2N)}(z_0)}{(2N)!}h^{2N}+\frac{f^{(3N)}(z_0)}{(3N)!}h^{3N}+\cdots\right] \end{aligned} \tag{2}$$

is also valid. Theorem 1 follows at once from (2), by well known properties of power series.

The special case $N=2$ of Theorem 1 is included in some recent results on real polynomials due to Anghelutza* and Whitney;† the latter writer also gives the corresponding special case of Theorem 2. Both Anghelutza and Whitney deal primarily with the study of *difference* equations, whereas the present properties involve the study of *q-difference* equations.

3. *Proof of Theorem* 2. Every polynomial $f(z)\equiv a_0z^{N-1}+a_1z^{N-2}+\cdots+a_{N-1}$ has the mean value property, for by (2) we may write for every z_0 and h

$$f(z_0+h)+f(z_0+\omega h)+\cdots+f(z_0+\omega^{N-1}h)\equiv Nf(z_0), \tag{3}$$

which is the mean value property. It is surprising that this simple property of polynomials (also a direct consequence of Lagrange's interpolation formula) is not well known; the writer

* Mathematica, vol. 6 (1932), pp. 1–7.

† This Bulletin, vol. 40 (1934), pp. 89–94.

knows of no explicit statement of it in the literature; the property is contained implicitly in a recent paper by J. H. Curtiss.*

Let $f(z)$ be a function analytic for all values of z, which has the mean value property. The fraction whose limit appears in the left-hand member of (1) is zero for every z_0 and for all values of h different from zero. Consequently we have $f^{(N)}(z) \equiv 0$, and $f(z)$ is a polynomial of degree $N-1$. Theorem 2 is established.

The trivial case $N=1$ is not excluded in Theorem 2; the mean value property requires $f(z_0) \equiv f(z_0+h)$, whence $f(z)$ is a constant.

It is worth remarking that the proof of $f^{(N)}(z) \equiv 0$ just given requires only that (3) should hold for real h; that is to say, it is sufficient if the mean value property holds with reference to all regular polygons with one horizontal radius. Moreover the proof that $f^{(N)}(z) \equiv 0$ requires only that (3) should hold for each $z=z_0$ for h sufficiently small; that is to say, it is sufficient if the mean value property holds for each z_0 with reference to regular polygons which are sufficiently small. Consequently if $f(z)$ is analytic in a region and has the mean value property with reference to all polygons contained in that region, then in that region $f(z)$ is a polynomial of degree $N-1$.

If a function is analytic in a region and possesses the mean value property for given N for all sufficiently small polgyons of N sides in that region, the function also possesses the mean value property for all sufficiently small polygons of $N+1$, $N+2, \cdots$ sides in that region.

Let a function $f(z)$ be analytic in the neighborhood of a *fixed* point z_0 and possess the mean value property merely with reference to all sufficiently small regular polygons of N sides *with center* z_0. It follows then from (3) and (2) that we have $0=f^{(N)}(z_0)=f^{(2N)}(z_0)=f^{(3N)}(z_0)= \cdots$. Reciprocally, if $f(z)$ is analytic in a circle whose center is a fixed point z_0 and if we have $0=f^{(N)}(z_0)=f^{(2N)}(z_0)=f^{(3N)}(z_0)= \cdots$, then it follows from (2) and (3) that $f(z)$ has the mean value property with reference to all regular polygons of N sides with center z_0 which lie in the given circle.

4. *Proof of Theorem* 3. Let $u(x, y)$ be a harmonic polynomial of degree $N-1$; the conjugate function $v(x, y)$ is also a har-

* Transactions of this Society, vol. 38 (1935), pp. 458–473; p. 462.

monic polynomial of degree $N-1$, and the analytic function $f(z) \equiv u(x, y)+iv(x, y)$ is a polynomial in z of degree $N-1$. The function $f(z)$ has the mean value property (3). If we take the real part of both members of (3) we obtain the mean value property for $u(x, y)$.

An arbitrary function $U(x, y)$ is said to have the *Gauss mean value property* if for every (x_0, y_0) the value $U(x_0, y_0)$ is the mean of the values of $U(x, y)$ over an arbitrary circumference whose center is (x_0, y_0). Every function harmonic in a region possesses the Gauss mean value property with respect to circumferences which together with their interiors lie in that region. Conversely [Koebe*], if a function $U(x, y)$ is continuous in a region and has the Gauss mean value property with respect to all sufficiently small circumferences in that region, then $U(x, y)$ is harmonic in that region.

Let now $u(x, y)$ be continuous in a region or throughout the entire plane, and possess the polygonal mean value property at least for sufficiently small polygons. Allow such a sufficiently small polygon to rotate rigidly about its center through an angle $2\pi/N$; the average of $u(x, y)$ over the entire circumscribed circle is the average of $u(x, y)$ at the N vertices averaged during this rotation, and is therefore equal to the value of $u(x, y)$ at the center of the circle. That is to say, the function $u(x, y)$ possesses the Gauss mean property, at least with respect to sufficiently small circumferences, and hence is harmonic.

Let $v(x, y)$ be conjugate to $u(x, y)$ (continuous and possessing the polygonal mean value property) in the given region or in a simply-connected subregion. Then $f(z)=u(x, y)+iv(x, y)$ is analytic there. Let us use equation (1) where h is chosen as real. At an arbitrary point z we have

$$f^{(N)}(z) = \frac{\partial^N u}{\partial x^N} + i\frac{\partial^N v}{\partial x^N}.$$

From the polygonal mean value property for $u(x, y)$ it follows by Theorem 1 with h real that $\partial^N u/\partial x^N$ (the real part of $f^{(N)}(z)$) vanishes identically, so the function $\partial^N v/\partial x^N$ conjugate to $\partial^N u/\partial x^N$ either vanishes identically or is identically a non-vanishing real constant. If $\partial^N v/\partial x^N$ vanishes identically, so also

* See for instance Kellogg, *Potential Theory*, 1929, pp. 224–228.

does $f^{(N)}(z)$, and Theorem 3 is established. If $\partial^N v/\partial x^N$ is identically a non-vanishing real constant q, we have $f^{(N)}(z) \equiv iq$,

$$f(z) \equiv u + iv \equiv \frac{iqz^N}{N!} + a_1 z^{N-1} + a_2 z^{N-2} + \cdots + a_N;$$

we shall reach a contradiction. Let $z = z_0$ be an arbitrary point of the plane or more generally of the region in which $u(x, y)$ is assumed to have the polygonal mean value property. We write

$$f(z) \equiv \frac{iq(z - z_0)^N}{N!} + b_1(z - z_0)^{N-1} + b_2(z - z_0)^{N-2} + \cdots + b^N.$$

The real part of $b_1(z-z_0)^{N-1}+b_2(z-z_0)^{N-2}+ \cdots +b_N$ is known to have the polygonal mean value property, hence the real part of $F(z) \equiv iq(z-z_0)^N/N!$ also has the polygonal mean value property; this is not true as we see by setting $z = z_0 + \alpha h$, where h is real and positive and where α is an Nth root of $-i$; we have

$$F(z_0 + \alpha h) + F(z_0 + \alpha\omega h) + F(z_0 + \alpha\omega^2 h) + \cdots + F(z_0 + \alpha\omega^{N-1}h) - NF(z_0) = qh^N/(N - 1)!,$$

whose real part is not zero. Thus $q \neq 0$ leads to a contradiction and Theorem 3 is completely established. We have indeed proved more than the italicized theorem, for we have shown that the polygonal mean value property at each point of a region for sufficiently small polygons implies that $u(x, y)$ is in that region a harmonic polynomial of degree $N-1$.

5. *Extensions of Theorem* 3. We have seen that the polygonal mean value property for sufficiently small polygons for a *continuous* function $u(x, y)$ implies that $u(x, y)$ is a harmonic polynomial of degree $N-1$. It is essential here to assume $u(x, y)$ continuous, as we see by the example

$$u(x, y) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0, \end{cases}$$

where N is even. At each point (x, y) the polygonal mean value property is possessed by this function for sufficiently small polygons. Yet the function is not a harmonic polynomial.

The continuity of $u(x, y)$ need not be assumed, however, if the polygons are not required to be small. More explicitly, let us indicate that *if* $u(x, y)$ (*not assumed continuous but measurable and Lebesgue integrable both on every circumference and interior to every circle in* R) *has the mean value property at all points of a region* R *with reference to all* N-*sided polygons in that region, then* $u(x, y)$ *is in that region a harmonic polynomial of degree* $N-1$. The polygonal mean value property yields the Gauss mean value property, as we have already seen. The Gauss mean value property implies by integration over the interior of any circle interior to R that the average of $u(x, y)$ over the interior of any such circle is the value of $u(x, y)$ at the center of the circle. The latter property implies continuity of $u(x, y)$. For let $P_0:(x_0, y_0)$ be an arbitrary point interior to R and let D in R be the interior of a circle whose center is (x_0, y_0) and radius r_0. Let $P_k:(x_k, y_k)$ be an arbitrary sequence of points approaching (x_0, y_0), and let D_k be (for k sufficiently large) the interior of a circle whose center is (x_k, y_k) and radius $r_0-\overline{P_0P_k}$. Then D_k lies interior to D; the radius of D_k approaches the radius of D; by Lebesgue's theorem the average value of $u(x, y)$ over D_k approaches the average value of $u(x, y)$ over D. Consequently $u(x, y)$ is continuous at the point (x_0, y_0). Theorem 3 extended now implies that $u(x, y)$ is in R a harmonic polynomial of degree $N-1$.

6. *Geometric Mean Values.* It is not without interest to compare our fundamental q-difference equation (3) with the analogous equation

$$\phi(z_0 + h)\phi(z_0 + \omega h) \cdots \phi(z_0 + \omega^{N-1}h) \equiv [\phi(z_0)]^N, \tag{4}$$

which expresses the condition that the value of the function $\phi(z)$ at a point z_0 is the geometric mean of the values at the vertices of a regular N-sided polygon whose center is z_0. Equation (4) reduces to (3) by taking the logarithms of both members of (4), so we have by Theorem 2 and its extension:

THEOREM 4. *A necessary and sufficient condition that an analytic function* $\phi(z)$ *have at every point* z_0 *of a region the mean value property* (4) *for all values of* h *or for all* h *sufficiently small is that* $\phi(z)$ *be identically zero or be of the form* $e^{p(z)}$, *where* $p(z)$ *is a polynomial of degree* $N-1$.

It will be noticed that it is possible to take logarithms in (4) in the neighborhood of a point z_0 if $\phi(z)$ is analytic, is not identically zero, and satisfies (4); if (4) is satisfied for every h sufficiently small we cannot have $\phi(z_0)=0$ unless at least one of the factors in the left-hand member of (4) vanishes for every h sufficiently small; in the latter case z_0 is a non-isolated zero of $\phi(z)$ and $\phi(z)$ vanishes identically.

Let now the real continuous function $U(x, y)$ have the property that its value at every point (x_0, y_0) is the geometric mean of its values at the vertices of every N-sided regular polygon whose center is (x_0, y_0). If $U(x, y)$ vanishes at a single point (x_1, y_1), it vanishes at every point, for we may choose a regular polygon of N sides whose center is (x_0, y_0) and which has a vertex at (x_1, y_1). Even if the mean value property is restricted to all polygons which are sufficiently small, an extension of the method of proof of Theorem 4 will yield the corresponding result:

THEOREM 5. *A necessary and sufficient condition that a real continuous function $U(x, y)$ have in a region or in the entire plane the property that the value at the center of every sufficiently small regular polygon of N sides is the geometric mean of the values at the vertices, is that $U(x, y)$ be identically zero or be of the form $\pm e^{u(x,y)}$ (the lower sign is permissible only if N is odd), where $u(x, y)$ is a harmonic polynomial of degree $N-1$.*

The trivial case $U(x, y)\equiv 0$ is henceforth excluded. Let now R be the given region, and let R' be one of the subregions of R obtained by deleting from R the points where $U(x, y)$ vanishes. The function $\log U(x, y)$ or $\log [-U(x, y)]$ (according as $U(x, y)$ is positive or negative in R') is continuous in R' and has the mean value property expressed by (3), so $U(x, y)$ is of the form $\pm e^{u(x,y)}$, where $u(x, y)$ is a harmonic polynomial of degree $N-1$. The function $u(x, y)$ cannot become negatively infinite at any finite boundary point of R', so $U(x, y)$ does not vanish at any finite boundary point of R', and R' may be chosen identical with R. Theorem 5 is established.

If the requirement of continuity is omitted, and if for each point only sufficiently small polygons are allowed, the conclusion is false, as we see by the illustration (N even)

$$U(x, y) = \begin{cases} e, & x > 0, \\ 1, & x = 0, \\ e^{-1}, & x < 0. \end{cases}$$

The analogs of Theorems 3 and 5 in three or more dimensions require methods other than those developed in the present note, and are postponed for another occasion.

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ON SEQUENCES OF INDEFINITE INTEGRALS

BY M. K. GOWURIN

1. *Introduction.* The chief result concerning the subject of this paper is due to Lebesgue* and can be formulated as follows:

If $\{f_n(t)\}$ *is a sequence of functions defined and integrable in* $J=(0, 1)$ *and if for every measurable set* $e \subset J$

$$\lim_n \int_e f_n(t)dt = 0, \tag{1}$$

then the sequence of indefinite integrals

$$\int_e f_n(t)dt \tag{2}$$

is uniformly absolutely continuous.

G. Fichtenholz† has shown that the same conclusion remains true, if the equality (1) is satisfied for all open sets e.

S. Saks‡ considered the space $R=\{x\}$ of the characteristic

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* *Sur les intégrales singulières,* Annales de Toulouse, (3), vol. 1 (1909), p. 58; see also Hahn, *Über Folgen linearen Operationen,* Monatshefte für Mathematik und Physik, vol. 32 (1922), p. 45.

† *Theory of simple definite integrals depending on a parameter,* Petrograd, 1918, p. 98 (in Russian) or *Sur les suites convergentes des intégrales définies,* Bulletin de l'Académie des Sciences de Pologne, Sér. A, Décembre, 1923, pp. 115, 117.

‡ *On some functionals,* Transactions of this Society, vol. 35 (1933), pp. 549–556.