

NOTE ON AN ASSOCIATIVE DISTRIBUTIVE
ALGEBRA IN WHICH THE COMMUTATIVE
LAW OF ADDITION DOES NOT HOLD

BY H. S. VANDIVER

1. *Introduction.* We shall give here a description of an algebra in which the elements are closed under the associative laws of addition and multiplication, a left and a right distributive law holds connecting addition and multiplication, and the usual equality axioms hold, yet the commutative law of addition does not hold.

2. *Certain Matrices.* As in other papers,* we define a semi-group as a set of elements closed under an associative operation, and a semi-ring as an algebra of the type above described. In the papers just cited we considered a set of elements

$$(1) \quad C_1, C_2, \dots, C_i, \dots, C_j; \quad (j \geq i),$$

with $C_{j+1} = C_i$ and forming a semi-ring, with

$$C_m + C_n = C_{m+n}, \quad C_m \cdot C_n = C_{mn}.$$

On page 581 of the first paper mentioned, we showed how to adjoin a zero element to this set, say C_0 .

In (1), set $i=j=1$, and consider the only two distinct elements in the set, after adjoining C_0 . We then have the relations

$$(2) \quad \begin{aligned} C_1 + C_1 &= C_1; & C_1 + C_0 &= C_0 + C_1 = C_1; & C_0 + C_0 &= C_0; \\ C_1 \cdot C_1 &= C_1; & C_0 \cdot C_1 &= C_0 \cdot C_1 = C_0. \end{aligned}$$

Note that these elements are *not* isomorphic to the two residue classes modulo 2, and the cancellation law of addition does not hold in general; that is, from $C_1 + C_0 = C_1 + C_1$, we cannot infer $C_0 = C_1$. Consider the matrices

$$A = \begin{pmatrix} C_0 & C_1 \\ C_0 & C_1 \end{pmatrix}; \quad B = \begin{pmatrix} C_1 & C_0 \\ C_1 & C_0 \end{pmatrix}; \quad C = \begin{pmatrix} C_1 & C_1 \\ C_1 & C_1 \end{pmatrix}.$$

* Proceedings of the National Academy of Sciences, vol. 20 (1934), p. 581; vol. 21 (1935), p. 162. This Bulletin, vol. 40 (1934), pp. 916-920.

Employing the usual definitions of addition and multiplication of matrices and (2), we have

$$(3) \quad \begin{aligned} A + B &= C; & A + A &= A; & B + B &= B; \\ C + C &= C; & A + C &= C; & B + C &= C. \end{aligned}$$

$$(4) \quad \begin{aligned} A^2 &= A; & B^2 &= B; & C^2 &= C; & AB &= B; & AC &= C; \\ BC &= C; & BA &= A; & CA &= A; & CB &= B. \end{aligned}$$

We define a *left identity* in a semi-group S as an element E_l such that $E_l M = M$ for any M in S , with an analogous definition of *right identity*. An element O_r such that $M O_r = O_r$ for each M in S , will be called a *right annihilator* (or annihilator) in S , and we define *left annihilator* in an analogous way. If an element is both a right and left annihilator it is called an *annihilator*. We note from (3) that A , B , and C form a semi-group under addition in which C is an annihilator. They form a semi-group under multiplication in which each element is a right-hand annihilator and a left-hand identity. By known properties of matrices, since C_0 and C_1 form a semi-ring, then A , B , and C generate a semi-ring. This result follows also from (3) and (4).

3. *Description of the System.* Consider a set of bracket symbols $[\alpha, \beta]$, where α and β are any of the matrices A , B , or C . We say that $[\alpha, \beta] = [\gamma, \delta]$ if and only if $\alpha = \gamma$, $\beta = \delta$. Hence there are nine distinct symbols. Also we define addition and multiplication as follows:

$$\begin{aligned} [\alpha, \beta] + [\gamma, \delta] &= [\alpha\gamma, \beta + \delta], \\ [\alpha, \beta] \cdot [\gamma, \delta] &= [\alpha\gamma, \beta\delta]. \end{aligned}$$

From these definitions we note that each element is a right annihilator and a left identity under multiplication. Hence the associative law of multiplication and the right and left distributive laws are easily seen to hold. The associative law of addition holds, since

$$\begin{aligned} ([\alpha, \beta] + [\gamma, \delta]) + [\epsilon, \theta] &= [\alpha\gamma, \beta + \delta] + [\epsilon, \theta] \\ &= [\alpha\gamma\epsilon, (\beta + \delta) + \theta] = [\alpha(\gamma\epsilon), \beta + (\delta + \theta)] \\ &= [\alpha, \beta] + ([\gamma, \delta] + [\epsilon, \theta]). \end{aligned}$$

Hence the bracket symbols form a semi-ring. The commutative law of addition does not hold in general in this semi-ring since

$$\begin{aligned} [A, \beta] + [B, \gamma] &= [AB, \beta + \gamma], \\ [B, \gamma] + [A, \beta] &= [BA, \gamma + \beta], \end{aligned}$$

but $BA \neq AB$. These symbols have a property under addition which might be called quasi-commutativity:

$$\begin{aligned} [\alpha, \beta] + [\alpha, \beta] + [\gamma, \delta] + [\gamma, \delta] \\ = [\alpha, \beta] + [\gamma, \delta] + [\alpha, \beta] + [\gamma, \delta], \end{aligned}$$

for the left-hand member reduces to $[\gamma, \beta + \beta + \delta + \delta]$ and the right to $[\gamma, \beta + \delta + \beta + \delta]$, which are equal since A , B , and C are commutative under addition. It is also easy to see that $MNMN = MMNN$, for M and N are bracket symbols.

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BRANCHED AND FOLDED COVERINGS*

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A simple example of a *branched* covering arises when one sphere is mapped on another so that each point of the first sphere goes into the point of the second which has the same latitude but double the longitude. This is a covering of degree two with simple branching at the north and south poles. As an example of a *folded* covering we take a torus, thought of as a sphere with a handle on one side, and project it radially inward on a smaller concentric sphere. The torus covers the sphere once but with a fold produced by collapse of the handle. The product of this torus-sphere covering with the previous sphere-sphere covering yields a torus-sphere covering of degree two which is both *branched and folded*. Suitable triangulation of the torus and the spheres will turn the above mappings into simplicial mappings in which each simplex maps barycentrically into a simplex of the same dimension. In what follows we make some rudimentary calculations concerning the branching and folding of a simplicial covering of one n -dimensional complex by

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