MILLER'S COLLECTED WORKS

The Collected Works of G. A. Miller. By H. R. Brahana, R. D. Carmichael, and A. B. Coble (the committee on publication), and G. A. Miller. University of Illinois Press. Volume 1, 1935. xi+475 pp.

In the preface written by the committee on publication, consisting of H. R. Brahana, R. D. Carmichael, and A. B. Coble, of the University of Illinois, it is stated that "the preparation of this volume is almost wholly the work of Professor Miller." This volume contains Miller's contributions up to the year 1900 together with three papers (numbered 1, 15, 62) written for the purpose of showing the place of this work in the theory and to give a brief survey of the historical development of the subject up to 1900.

The papers in this volume with exception of 1, 15, and 62, are arranged substantially in the order in which they were published, but for the purpose of exhibiting the nature of the work contained in them it will be convenient to group them under the following headings: (1) Papers which relate to the determination of groups of finite order. (2) Expository papers on the construction of permutation groups; applications. (3) Papers which deal with the theory of groups of finite order. (4) Paper on recent progress; miscellaneous papers. These will be considered in the order given.

(1) *Papers which relate to the determination of groups of finite order.*

These are as follows: (a) Permutation groups of given degree; (b) Abstract groups of given order; and (c) Non-regular permutation groups of given order.

(a) *Permutation groups of given degree.*

To give a proper introduction and setting to his early papers on the listing of permutation groups of given degrees, Miller has written a special paper (numbered 1) entitled *Historical note on the determination of all permutation groups of low degrees.* In this paper there is given a clear account of enumerations made by earlier writers along with a brief statement of the methods used.

Including the historical note just mentioned there are fifteen papers devoted to the listing of permutation groups of given degrees. The most important of these papers are devoted to the determination of all permutation groups through degree ten, to the determination of all transitive groups of degrees twelve, thirteen, and fourteen, and to the finding of all primitive groups of degrees fifteen, sixteen, and seventeen. Lists of the intransitive groups of degree ten and of the imprimitive groups of degrees thirteen and fourteen were first published by Miller. Other papers are concerned with the correction of errors in lists published by earlier writers. Errors were found in all degrees from eight to seventeen, inclusive, with the exception of degrees eleven, thirteen, and fourteen. To find these errors it was necessary to make complete lists for all degrees examined by earlier writers, and in the early stages when the theory upon which this work is based was not far developed, this involved a considerable amount of tentative work. Later determinations, which were based wholly or in large part on theory, show that his earlier com-

putations were entirely accurate. His important memoir on the substitution groups whose degrees do not exceed eight, and his determination of the primitive permutation groups of degree ten are based entirely on theoretical considerations. For his work on the permutation groups of degree ten he was awarded in 1900 a prize by the Cracow Academy of Sciences.

(b) *Abstract groups of given order.*

To provide an introduction to his early papers on abstract groups, Miller has written a special paper (numbered 15) entitled *Historical note on the determination of abstract groups of given order.* This note is an amplification of a similar paper published in the Journal of the Indian Mathematical Society in 1932. In this paper there is given some account of postulational definitions of finite abstract groups, and a sufficient account of the work that has been done on the determination of abstract groups of given order to afford a proper setting for the author's work in this field. It also gives a brief statement of the work by the author on a problem initiated by him, namely, to determine all orders for which a given number of abstract groups exist.

In addition to the historical note just referred to, there are five papers in this volume devoted to the determination of all abstract groups of given order. One of these gives a complete determination of all abstract groups whose orders are less than forty-eight. In this paper the groups of orders sixteen, twenty-four, thirty-two, and forty are treated in some detail. The groups of other orders less than forty-eight follow readily from earlier considerations. The paper includes also some remarks on regular groups and the earliest published development of the properties of commutator sub-groups, the usefulness of which was widely recognized later.

The abstract groups of order $8p$ where p is any prime number are determined in a second paper. In this paper consideration is given first to those groups which contain an invariant sub-group of order p, and second to those groups which do not contain such a group. In this and the preceding paper the method used is that of constructing imprimitive permutation groups of given degree. A third paper is devoted to the determination of all abstract groups whose orders are greater than forty-seven and less than sixty-four, and of those whose orders are $2p^3$ where p is any prime number. In this paper the groups of order forty-eight are given particular attention and those of order fifty-four are obtained as a special case of the groups of order $2p^3$. The other groups whose orders are less than sixty-four and greater than forty-seven follow readily from determinations made by earlier writers. Other papers which relate to abstract groups of given orders consist largely of brief reports published in the Comptes Rendus on lists of these groups. One of these gives also some information on permutation groups whose degrees are less than thirteen.

(c) *Non-regular permutation groups of given order.*

In addition to the papers which relate to the determination of all permutation groups of given degrees and of all abstract groups of given orders, Miller has written several papers which relate to the determination of permutation groups of given orders and to the number of transitive permutation groups which are isomorphic to a given group. In this volume there is one paper on the permutation groups of order four and another on the permutation groups

whose orders are the product of two distinct prime numbers. In other papers determinations of the non-regular transitive permutation groups of orders p^3, pqr, and $8p$ are made, where p, q, r are prime numbers. Two papers treat the number of transitive permutation groups isomorphic to a given group. One of these relates to the transitive permutation groups simply isomorphic to the symmetric or to the alternating group of degree six; the other to the number of transitive permutation groups that have a 1, ∞ isomorphism to a given group.

(2) *Expository papers on the construction of permutation groups; applications.*

There is included in this volume a series of three expository papers (numbered 9, 10, 11) which were published in volumes 2 and 3 of the American Mathematical Monthly under the titles: *Remarks on substitution groups, Introduction to substitution groups,* and *Applications of substitution groups.* The first of these presents a few basic concepts in the theory of substitution groups and lists the substitution groups of degrees two and three. The second paper develops in a simple manner the elementary methods of constructing permutation groups and applies these methods to the construction of all permutation groups whose degrees do not exceed six. This paper is quite valuable since it gives in simple form what is perhaps the best discussion to be found in the literature of methods for finding permutation groups of given small degrees. The third paper is devoted to the study of some of the relations which exist between substitution groups and functions that involve a given number of variables. Applications are then made to determine the number of different formal values of the cross-ratios of four points under the symmetric group of degree four, to find the other formulas in trigonometry which correspond to that for the sine of a given half-angle of a triangle in terms of the sides, and to a discussion of a simple problem in factoring. Another paper published in the Monthly is devoted to a practical application of a substitution group in spherical trigonometry. Three additional papers also published in the Monthly, which are largely expository in character, treat the groups determined by a given group, a method for constructing intransitive permutation groups, and the presentation in a very simple manner of some elements of substitution groups.

(3) *Papers which deal with the theory of groups of finite order.*

These papers may be classified into those which relate to the properties of abstract groups and those which relate to the properties of permutation groups. Those on abstract groups deal, for the most part, with properties of commutators and commutator groups, Hamilton groups, perfect groups, and the quaternion group. Important properties of each of these special classes of groups are developed by Miller. His work on commutator sub-groups merits special attention since the earliest published properties of these groups were given in his paper on the regular substitution groups whose orders are less than forty-eight which was published in the Quarterly Journal of Mathematics, volume 28 (1896). In this paper application of these properties was made to test the solvability of the groups found. Two later papers are devoted to further study of commutators and commutator sub-groups. Other papers devoted to abstract groups deal with an extension of Sylow's Theorem and with properties of groups of order p^∞ where p is a prime number.

Several general theorems on permutation groups are proved by Miller in

his papers which relate to lists of permutation groups of given degrees and of abstract groups of given orders. For example, his paper in volume 28 (1897) of Proceedings of the London Mathematical Society on primitive substitution groups of degree fifteen contains the proofs of several theorems which relate to the maximal sub-group of degree $n-1$ in a simply transitive primitive group of degree n. Additional properties of these sub-groups are developed in a brief paper published later in this Bulletin. Two papers on permutation groups determine conditions under which permutation groups of given special degrees contain simple groups. Two additional papers are devoted to multiply transitive groups. One of these includes the proof of a theorem on an upper limit of transitivity and the other gives a necessary and sufficient condition that a group may be represented as a multiply transitive group. Other papers which relate to permutation groups are somewhat more elementary in character. One of these gives a new proof of the well known theorem on the average number of letters in a permutation group of degree n; a second relates to the matter of definition of distinct permutation groups; and a third deals with the simple isomorphisms of a group to itself.

(4) *Paper on recent progress; miscellaneous papers.*

A paper entitled *Report on recent progress in the theory of groups of a finite order* was published in Volume 5 (1899) of this Bulletin. It presents developments of importance in this field during the five preceding years which were not fully treated in existing treatises. The developments are presented under the headings solvable groups, simple groups, substitution groups, and abstract groups. Attention is called also to unsolved problems of interest.

Miscellaneous papers not included in what precedes present Sophus Lie's views on several important points in modern mathematics, some reminiscences in regard to Sophus Lie, several points in the theory of groups of a finite order (this being a paper read before the Oliver Mathematics Club, Cornell University), and a note on Burnside's *Theory of Groups*. In the last note attention was called to several errors in the first edition of this treatise.

The last paper in the volume (numbered 62) is entitled *History of the theory of groups up to* 1900. Along with the other two papers (numbered 1 and 15) written for this volume by Miller, it shows the connection between his work before 1900 and the general developments in the subject, and it gives a suitable basis for seeing the connection between these early papers and his later work.

In this paper the history of the theory of groups up to 1900 is divided into five periods, as follows: (1) Developments antedating the beginning of the nineteenth century. (2) The first forty years of the nineteenth century. (3) The period from 1840 to 1860. (4) The period from 1860 to 1880. (5) The years 1880 to 1900. Excellent brief accounts are given of all important developments in group theory during each of these five periods with the exception of the important work of G. Frobenius toward the end of the last period on group characters. Reference to this work of Frobenius is given, however, in the paper (numbered 48) on recent progress. Much valuable information is also given on the influences in the various countries which led to these developments. This important paper will be of value to many students of group theory.

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