the graph and its asymptote would clearly give the value of ν. If, on the other hand, the graph in question shows no asymptote, that fact signifies that $\mu = 1$. In that case the left member of (13) is to be plotted. Since ν is now not equal to unity this graph will show an asymptote, the slope and intercept of which determine h and ν.

HARVARD UNIVERSITY

A SERIES OF INVOLUTORIAL CREMONA SPACE TRANSFORMATIONS DEFINED BY A PENCIL OF RULED CUBIC SURFACES*

BY AMOS BLACK

1. *Introduction.* A series of involutorial Cremona transformations of space were defined by Snyder† by means of a correspondence between the surfaces of a pencil of ruled surfaces and the points of a rational curve, called the director curve. The director curve was a part of the basis of the pencil and two of the chief characteristics of the transformation were: one of the principal surfaces was a ruled surface R, all of whose generators were parasitic lines; all the tangent planes of the surfaces of the homaloidal web along a certain curve were fixed, being determined by the surface R.

In this paper we shall define a series of transformations by means of a correspondence between the surfaces of a pencil of ruled cubic surfaces and the points of certain rational curves. The director curve is not part of the basis of the pencil; one of the principal surfaces is a ruled surface R, all of whose generators are parasitic lines; all the surfaces of the homaloidal web have fixed tangent planes along a certain curve, but none of the fixed planes are determined by R.

2. *Definition of the Transformation.* Given a pencil of ruled cubic surfaces $|F_3|$ whose basis curve consists of a double line d

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and a rational space quintic curve δ_5 which has d for a quadrisecant; also, given a rational space curve r_m, of order m, which meets d and δ_5 in $m-1$ and $m+1$ points, respectively; then a general surface F_3 of the pencil meets r_m in $3m-1$ fixed points and in just one general point O. Thus there is a $(1, 1)$ correspondence between the surfaces of $|F_3|$ and the points O of r_m. We define $F_3:O$ and O as associated surface and point.

A point $P(y)$ in space will determine a unique surface F_3 of the pencil, hence a unique point $O(z)$ on r_m. The line PO meets F_3 in $P, O,$ and one other point $P'(x)$, which is defined as the image of P. Then on each line of the complex of lines meeting r_m lies a pair of points P, P' in involution.

We shall discuss the case where $m=2$ in detail, and then indicate results for m general.

3. *Equations of the Transformation.* Let the equations of d and r_2 be

(1) $$d: \quad x_3 = 0, \quad x_4 = 0,$$

(2) $$r_2: \quad x_1 x_4 - x_3{}^2 = 0, \quad x_2 = 0.$$

Then $D = [r_2, d] = (1, 0, 0, 0)$, and any point on r_2 can be represented by

(3) $$O(z) = O(z_1, z_2, z_3, z_4) = (\lambda^2, 0, \lambda\mu, \mu^2).$$

Let the pencil of surfaces $|F_3| : d^2\delta_5$ be

(4) $$\mu F(x) - \lambda F'(x) = 0,$$

where λ, μ in (4) is the same as in (3). If $F(O):O$, then

(5)
$$
\begin{aligned}
F(x) = & \; [a_3' x_1 + a_2 x_2 + b_1' x_3 + (b_3' + a_4' - b_1)x_4]x_3{}^2 \\
& + [b_1 x_1 + b_2 x_2 + (b_4' + c_1')x_3 + b_4 x_4]x_3 x_4 \\
& + [c_1 x_1 + c_2 x_2 + c_4' x_3]x_4{}^2, \\
F'(x) = & \; [a_2' x_2 + a_3' x_3 + a_4' x_4]x_3{}^2 \\
& + [b_1' x_1 + b_2' x_2 + b_3' x_3 + b_4' x_4]x_3 x_4 \\
& + [c_1' x_1 + c_2' x_2 + (c_1 + b_4)x_3 + c_4' x_4]x_4{}^2.
\end{aligned}
$$

Denote by A, B, C and A', B', C' the coefficients (containing x_1 and x_2 only) of $x_3{}^2$, $x_3 x_4$, $x_4{}^2$ in $F(x)$ and $F'(x)$, respectively, (that is, $A = a_3' x_1 + a_2 x_2$); and by α, β, γ the determinants

$$|BC'| = BC' - B'C, \quad |AC'|, \quad |AB'|;$$

and let

$$\bar{\mu} = A'x_3^2 + B'x_3x_4 + C'x_4^2.$$

Any point on the line joining $P(y)$ to $O(z)$ has coordinates

(6) $$x_i = \rho y_i + \sigma z_i, \qquad (i = 1, 2, 3, 4).$$

The value of ρ/σ for $P'(x)$ is given by

(7) $$\mu F(\rho y + \sigma z) - \lambda F'(\rho y + \sigma z) = 0.$$

Since P is on (4) and O is on r_m, we find

(8) $$\rho[\mu F(z, y) - \lambda F'(z, y)] + \sigma[\mu F(y, z) - \lambda F'(y, z)] = 0,$$

where $F(y, z)$, $F'(y, z)$ are the first polars of $F(y)$, $F'(y)$ with respect to (z), and $F(z, y)$, $F'(z, y)$ are the first polars of $F(z)$, $F'(z)$ with respect to (y), and $\lambda/\mu = F(y)/F'(y)$. Hence

$$\frac{\rho}{\sigma} = -\frac{L_{16}}{K_{11}},$$

where

(9) $$\begin{aligned} L_{16} &= F'(y)F(y, z) - F(y)F'(y, z), \\ K_{11} &= F'(y)F(z, y) - F(y)F'(z, y). \end{aligned}$$

However L_{16} and K_{11} are factorable, and

(10) $$K_{11} = R_4K_7, \qquad L_{16} = R_4F'L_9,$$

where $F' = F'(y)$ and

(11) $$\begin{aligned} R_4 &= y_4F - y_3F', \quad K_7 = \bar{\mu}(\gamma y_3^2 + 2\beta y_3y_4 + \alpha y_4^2) + F_7(y), \\ L_9 &= \bar{\mu}\bar{L}_6 + F_9(y), \\ \bar{L}_6 &= (A\gamma + A'\beta)y_3^3 + (A\beta + A'\alpha + B\gamma + B'\beta)y_3^2y_4 \\ &\quad + (B\beta + B'\alpha + C\gamma + C'\beta)y_3y_4^2 + (C\beta + C'\alpha)y_4^3, \end{aligned}$$

and where $F_7(y)$, $F_9(y)$ are terms containing y_3, y_4 to higher degree than 4 and 5, respectively. On removing the factor R_4, the involutorial transformation is therefore expressed by

(12) $$I_{13}: \quad x_i = y_iF'L_9 - K_7z_i, \qquad (i = 1, 2, 3, 4).$$

It is evident from equations (12) that $K_7 = 0$ is the surface

of invariant points, that $L_9 = 0$ is the image of r_2, and that $F' = 0$ is the image of D.

4. *Images of the Fundamental Elements.* Through a general point O of r_2 passes a generator g of $F(O)$. If P is any point on g, since PO lies on $F(O)$, then P' is indeterminate, and the line g is parasitic. As O traces r_2, the line g generates the surface $R_4 = 0$. Since all the generators of R_4 are parasitic lines, the surface should appear as a factor of the transformation, as we have seen from (10).

Consider a general point P on d. It remains invariant on every $F(O)$ of the pencil, except when PO is a generator g_O on $F(O)$, hence a generator of R_4. Thus the total image of d is R_4.

Since D is a double point on $F(D) \equiv F'$, every point of F' is transformed into D, hence $D \sim F'$.

The tangent plane of $F(O)$ at O intersects $F(O)$ in a generator g_O and a conic $c_2:O$. Every point of c_2 is transformed into O, hence $O \sim c_2$. As O traces r_2, the c_2 generates the surface $L_9 = 0$. The point O is invariant in the direction of the tangent of c_2, hence L_9 and K_7 are tangent along r_2.

On every generator of the rational cone with vertex P on δ_5 and standing on r_2 is one point P', the image of P. The locus of P' is a curve c of order two increased by the number of times PO is tangent to $F(O)$ at P. Given an arbitrary point O on r_2, the tangent plane of $F(O)$ at P cuts r_2 in two points M. Given a point M, there is a unique surface for which MP is a tangent at P, thus one point O. This (1, 2) correspondence on r_2 has three coincidences, hence $P \sim c_5:P^3$. However, through P passes one generator g_O of $F(O)$. Hence c_5 is composite and consists of g_O and a $c_4:P^2$. As P traces δ_5 the g_O generates R_4, and c_4 generates a surface $\Delta_{22} = 0$. Since R_4 factors out of the transformation, the total image of δ_5 is Δ_{22}. The equations of Δ_{22} are found by finding the image of $F(x)$. We see that

$$F(x):d^2D^2\delta_5 \sim FR_4^2F'^2\Delta_{22},$$

where

(13) $$\Delta_{22} = \bar{\mu}^4\bar{L}_6(\beta^2 - \alpha\gamma) + F_{22}(y),$$

and $F_{22}(y)$ contains y_3, y_4 to degree higher than eleven. The point P is invariant in the directions of the two tangents of $c_4:P^2$,

hence two sheets of Δ_{22} are tangent to the two sheets of K_7 along δ_5.

5. *Contact Along d.* The two tangent planes of $F'=0$ at any point on d are identical with those of $\bar{\mu}=0$. It is evident from equations (11), (12), and (13) that each tangent plane of $\bar{\mu}$ (hence of F') is the tangent plane of one sheet of $K_7=0$, one sheet of $L_9=0$, two sheets of a general homaloid $S_{13}=0$, and four sheets of $\Delta_{22}=0$ at every point on d. Also the remaining three sheets of L_9, S_{13}, and Δ_{22} have fixed tangent planes along d, the three planes being identical with the tangent planes of $\bar{L}_6$ along d. The image due to the former contact is F' and the latter is R_4.

Although every tangent plane of L_9, S_{13}, and Δ_{22} is fixed at any point on d, they are all distinct from the tangent planes of R_4 at the point. The tangent planes of F' at D are

$$y_4 = 0, \quad \pi = b_1' y_3 + c_1' y_4 = 0.$$

At D, one of the three tangent planes of L_9, S_{13}, Δ_{22} coincides with π.

6. *Determination of the Parasitic Lines.* We have seen that every generator of R_4 is a parasitic line. In general none of these generators lies on any other surface of the transformation. We wish to find which of these lines do lie on other surfaces, and any other parasitic lines which may arise.

Given a point O on r_2, the generator g_O on $F(O)$ intersects d in a point L. The surface $F(O)$ has two pinch points M on d. If a point L and a point M coincide, then a generator g_O on $F(O)$ passes through a pinch point of $F(O)$. In this case g_O is not only parasitic but is a line of contact for S_{13}, L_9, K_7, R_4, Δ_{22}.

Given a point L there are two points M. Given a point M, there are two surfaces for which M is a pinch point, hence two points L. This $(2, 2)$ symmetric correspondence on d has two coincidences, hence there are two such parasitic lines. Denote them by $2g$.

The surface F' has two generators passing through D, its associated point. Each is a parasitic line. One lies in the plane $y_4 = 0$, hence lies on R_4, and factors out of the transformation. The other lies in the plane π and is a parasitic line for the transformation. This line g' is simple on S_{13}, K_7, F', and Δ_{22}.

Since δ_5 is rational and has d for quadrisecant, the surface of trisecants is a ruled surface of order eight, R_8: $\delta_5{}^3 d^4$. Since r_2 meets d once and δ_5 three times there are three trisecants of δ_5 which meet r_2. If O_1 is a point on r_2 through which passes a trisecant of δ_5, then the trisecant lies on $F(O_1)$ and is parasitic. Thus there are three such parasitic lines, $3g''$. They are simple on S_{13}, L_9, K_7 and triple on Δ_{22}.

7. *Table of Characteristics.* The images of planes and of fundamental elements can now be expressed by the following table:

$$S_1 \sim S_{13} : r_2{}^2 d^{7+2t+2t+3t'} D^7 \delta_5{}^4 2g\ g'3g'',$$

$$r_2 \sim L_9 : r_2 d^{5+t+t+3t'} D^5 \delta_5{}^3 2g3g'',$$

$$d \sim R_4 : r_2 d^3 D^3 \delta_5 2g,$$

$$D \sim F' : d^{2+t+t} D^2 \delta_5 g',$$

$$\delta_5 \sim \Delta_{22} : r_2{}^4 d^{11+4t+4t+3t'} D^{11} \delta_5{}^7 2g\ g'3g''^3,$$

$$K_7 : r_2 d^{4+t+t} D^4 \delta_5{}^2 2g\ g'3g'',$$

$$J_{48} = L_9 R_4{}^2 F'^3 \Delta_{22},$$

where the t and t' in the multiplicity of d stands for the contact as given in §5.

Thus the complete intersection of two surfaces of the web may be written

$$[S_{13}, S_{13}'] = C_{13} r_2{}^4 d^{49+4+4+3} \delta_5{}^{16} 2(2g)g'3g''.$$

8. *The Transformation for m General.* We replace r_2 by a rational curve r_m, which meets d and δ_5 in $m-1$ and $m+1$ points, respectively, with parametric representation

$$(14) \qquad\qquad z_i = z_i(\lambda, \mu), \qquad\qquad (i = 1, 2, 3, 4),$$

where the z_i are homogeneous functions of degree m in λ, μ. Denote the $m-1$ points of intersection of r_m with d by D_i, and the associated surfaces by $F^{(i)}$. The transformation has the form

$$(15) \qquad I_{6m+1} : x_i = y_i F^{(1)} F^{(2)} \cdots F^{(m-1)} L_{3m+3} - K_{3m+1} z_i(F, F'),$$

$$(i = 1, 2, 3, 4),$$

where $K_{3m+1} = 0$ is the surface of invariant points, $L_{3m+3} = 0$ is the image of r_m, and $F^{(i)} = 0$ is the image of D_i. The image of d is $R_4 = 0$, as before. The image of a general point P on δ_5 is a curve

$c_{2m}:P^m$. As P traces δ_5, the c_{2m} generates a surface Δ_{12m-2}, the total image of δ_5.

Each of the two tangent planes of each of the $m-1$ surfaces $F^{(i)}$ at each point of d is the tangent plane of one sheet of L_{3m+3}, one sheet of K_{3m+1}, two sheets of a general S_{6m+1}, and four sheets of Δ_{12m-2}. Also the remaining three sheets of L_{3m+3}, S_{6m+1}, and Δ_{12m-2} have fixed tangent planes along d.

The parasitic lines are as follows: $2g$, two generators of R_4 which are simple lines of contact for L, K, S, R, Δ; $(m-1)g'$, $m-1$ lines, one on each $F^{(i)}$, and simple on S, K, Δ; $(m+1)g''$, trisecants of δ_5 meeting r_m, simple on S, L, K and triple on Δ.

The results can be expressed by the following table:

$$S_1 \sim S_{6m+1}:r_m^2 d^{4m-1+2t_i+2t_i+3t'}D_i^{4m-1}\delta_5^{2m}2g(m-1)g'(m+1)g'',$$

$$r_m = L_{3m+3}:r_m d^{2m+1+t_i+t_i+3t'}D_i^{2m+1}\delta_5^{m+1}2g(m+1)g'',$$

$$d \sim R_4:r_m d^3 D_i^3 \delta_5 2g,$$

$$D_j \sim F^{(j)}:d^{2+t_j+t_i}D_j^2 \delta_5 g_j',$$

$$\delta_5 \sim \Delta_{12m-2}:r_m^4 d^{8m-5+4t_i+4t_i+3t'}D_i^{8m-5}\delta_5^{4m-1}2g(m-1)g'(m+1)g''^3$$

$$K_{3m+1}:r_m d^{2m+t_i+t_i}D_i^{2m}\delta_5 2g(m-1)g'(m+1)g'',$$

$$J_{24m} = L_{3m+3}R_4^2 F^{(1)^3} \cdots F^{(m-1)^3}\Delta_{12m-2},$$

where the t_i and t' represent the contact along d, and $i = 1, 2, \cdots, m-1$.

The complete intersection of two surfaces of the web is

$$[S_{6m+1}, S'_{6m+1}]$$
$$= C_{6m+1}r_2^4 d^{(4m-1)^2+(m-1)(4+4)+3}\delta_5^{(2m)^2}2(2g)(m-1)g'(m+1)g''.$$

CORNELL UNIVERSITY