ON THE NON-VANISHING OF THE JACOBIAN IN CERTAIN ONE-TO-ONE MAPPINGS

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THEOREM 1. *If $u(x, y)$ and $v(x, y)$ are harmonic, $u(0, 0) = v(0, 0) = 0$, and if there exists a neighborhood N_1 of the origin of the xy plane and a neighborhood N_2 of the origin of the uv plane such that $u(x, y)$ and $v(x, y)$ establish a mapping of N_1 onto N_2 which is one-to-one both ways, then the Jacobian $\partial(u, v)/\partial(x, y)$ does not vanish at the origin.*

PROOF. As the statement of Theorem 1 remains invariant under homogeneous linear transformations of the *uv* plane, we may assume, in the developments in polar coordinates for u and v, that

$$u = \sum_{i}^{\infty} [a_n r^n \cos n\theta + b_n r^n \sin n\theta], \qquad (a_i^2 + b_i^2 \neq 0),$$

$$v = \sum_{k}^{\infty} [A_n r^n \cos n\theta + B_n r^n \sin n\theta], \qquad (A_k^2 + B_k^2 \neq 0),$$

that the positive index i does not exceed k, and that for $i=k$ we have $a_k B_k - A_k b_k \neq 0$. Considering the case $i=k$ first, we may, because of the invariance mentioned, assume

$$a_k = B_k = 1, \qquad b_k = A_k = 0.$$

For small values of r, the auxiliary mapping,

$$\bar{u} = r^k \cos k\theta, \qquad \bar{v} = r^k \sin k\theta,$$

can be continuously joined with the given one for $0 \leqq t \leqq 1$ by

$$u_t = r^k \cos k\theta + t \sum_{k+1}^{\infty} (a_n r^n \cos n\theta + b_n r^n \sin n\theta),$$

$$v_t = r^k \sin k\theta + t \sum_{k+1}^{\infty} (A_n r^n \cos n\theta + B_n r^n \sin n\theta).$$

The vector (u_t, v_t) thereby never differs by more than a vector of length $r^k/2$ from the vector $(\bar{u}, \bar{v})$ whose length is r^k. Thus the index of the origin* in both fields (u, v) and $(\bar{u}, \bar{v})$ is the same, and as it is ± 1 for one-to-one mappings we conclude $i=k=1$ and our theorem follows.

In the remaining case $i<k$ we consider the Jacobian

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \frac{1}{r}\,\frac{\partial(u, v)}{\partial(r, \theta)}.$$

We find by elementary computation that the development of J starts with the terms of lowest degree in r

$$kir^{i+k-1}\left[(A_k a_i + B_k b_i)\sin(i-k)\theta + (a_i B_k - b_i A_k)\cos(i-k)\theta\right]$$

or, with a suitable angle θ_0, with the term

$$kir^{i+k-1}(A_k^2 + B_k^2)^{1/2}(a_i^2 + b_i^2)^{1/2}\cos\left[(i-k)\theta - \theta_0\right].$$

Hence, for sufficiently small values of r, J assumes both positive and negative values, while in a one-to-one map the Jacobian, where it does not vanish, is of the same sign as the index of the map which is either $+1$ everywhere or -1 everywhere. Hence $i<k$ is impossible and our proof is completed.

THEOREM 2. *Let $u(x, y)$ and $v(x, y)$ be analytic functions of x and y in a neighborhood N_1 of the origin of the xy plane which they map onto a neighborhood N_2 of the origin of the uv plane in a one-to-one correspondence. Suppose, moreover, that u and v are solutions of the following equations*:

$$(1)\quad \begin{aligned}&\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + a\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + b\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\\ &\qquad + c\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + d\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) = 0,\end{aligned}$$

$$(2)\quad \begin{aligned}&\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + A\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + B\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\\ &\qquad + C\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + D\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) = 0,\end{aligned}$$

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* For the topological notions used, see W. Fenchel, *Elementare Beweise und Anwendungen einiger Fixpunktsätze*, Matematisk Tidsskrift, (B), 1932, p. 66.

in which $a(u, v)$, $b(u, v)$, $\cdots$, $D(u, v)$ are analytic functions of u and v defined for (u, v) in N_2. Then the Jacobian $\partial(u, v)/\partial(x, y)$ does not vanish at the origin.

PROOF. Suppose, without loss of generality, that the power series for u and v in x and y start with non-vanishing terms of ith and kth degree, respectively, and that furthermore $i \leqq k$, $i > 0$. Then the terms of ith degree in $u(x, y)$ form a harmonic function, because this is trivial for $i = 1$, and for $i > 1$ the only terms in the development of the left hand of (1) of degree $i-2$ are furnished by those of $\partial^2u/\partial x^2 + \partial^2 u/\partial y^2$. Introduce $V(x, y) = v(x, y) - F(u)$, where F is analytic in u and $F(0) = 0$. Instead of (1) and (2) we find

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + [a(u, V + F(u)) + F'(u)b(u, V + F(u)) \\
&\qquad + F'^2(u)c(u, V + F(u))]\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + \cdots = 0,
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
&\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + F'(u)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\
&\quad + [A(u, V + F(u)) + F'(u)B(u, V + F) + F'^2(u)C(u, V + F) \\
&\qquad + F''(u)]\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + \cdots = 0,
\end{aligned} \tag{4}
$$

and, by substitution of (3) in (4),

$$
\begin{aligned}
&\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + [A(u, V + F) + F'B + F'^2C + F'' - F'a \\
&\qquad - F'^2b - F'^3c]\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + \cdots = 0,
\end{aligned} \tag{5}
$$

where the omitted terms are linear in

$$
\frac{\partial u}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial V}{\partial y}, \quad \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right), \quad \text{and} \quad \frac{\partial(u, V)}{\partial(x, y)}.
$$

Now notice that V vanishes for $x = y = 0$ and that in its development the lowest degree K of non-vanishing terms still is $\geqq i$.

Let us distinguish two cases, $k > i$ and $k = i$. For $k > i$, we construct a function $F(u)$, solution of the ordinary differential equation of second order

$$
\begin{aligned}
(6) \qquad & A(u, F(u)) + F'(u)B(u, F(u)) \\
& \qquad\qquad + F'^2C - F'a - F'^2b - F'^3c + F'' = 0,
\end{aligned}
$$

vanishing with its first derivative for $u=0$. We have $k>i$, as $F(u)$, considered as function of x and y starts with terms of degree $\geqq 2i$. Equation (5) shows now a coefficient of $(\partial u/\partial x)^2+(\partial u/\partial y)^2$ whose development in u and V has *all terms divisible* by V. Thus the x, y series for the left hand of (5) obtains all terms of degree $K-2$ from the expression $\partial^2 V/\partial x^2+\partial^2 V/\partial y^2$, whence we conclude that in V the terms of lowest order K form a harmonic function. As in the proof of Theorem 1, we conclude that the Jacobian $\partial(u, V)/\partial(x, y)$, which equals $\partial(u, v)/\partial(x, y)$, would assume both positive and negative values in every neighborhood of the origin, which leads to a contradiction. Thus $i<k$ is impossible.

In the case $i=k$ we again choose $F(u)$ as solution of (6), vanishing for $u=0$, but we try to determine $F'(u)$ in such a way that k results $>i$. If this were possible, we could apply the same argument as at the end of the last paragraph, and find the same contradiction. Hence, should $i=k=1$, we would have the desired inequality

$$\frac{\partial(u, V)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(x, y)} \neq 0$$

at the origin. If, however, $i=k>1$, we may determine the linear term of $F(u)$ such that for polar coordinates in the xy plane and suitable determination of $\theta=0$ we have

$$
\begin{aligned}
u &= \text{const. } r^k \cos k\theta + r^{k+1}(\cdots), \\
V &= \text{const. } r^k \sin k\theta + r^{k+1}(\cdots).
\end{aligned}
$$

This leads, by the same reasoning as in the proof of Theorem 1, to the conclusion that the index of the vector field (u, V) at the origin is $k\neq 1$. But together with the functions $u(x, y)$, $v(x, y)$, the functions $u(x, y)$, $V(x, y)=v(x, y)-F(u)$ also establish a one-to-one mapping of the xy plane, because the correspondence between the uv plane and the uV plane evidently is one-to-one. Hence the index ought to be ± 1, which completes our proof.

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