ON THE PRINCIPLES OF HAMILTON AND CARTAN
(SUPPLEMENTARY NOTE)

BY J. W. CAMPBELL

A request from Dr. D. C. Lewis for an explanation regarding a transformation which appeared in my paper on this topic (this Bulletin, February, 1936) led me on examination to find that the results there given are true in a restricted case but not in general. They are valid when the a_{rs} and the a_s of the non-holonomic relations are functions of t only. Also, under these conditions it can be shown that the order of integration in (15) and (16) is reversible and that then the action integral

$$\int_{u_0}^{u_1} \sum_1^k p_r dq_r - H dt$$

is an extremal for an arc of a trajectory as compared with all neighboring paths which are kinematically possible.

There is, however, a coordination of the principles for the general case. For, using the notation which was adopted, consider the expression

$$\begin{aligned}\sum_1^k \Bigg\{ \left(dq_r - \frac{\partial H}{\partial p_r} dt \right) \delta p_r & \\ + \left(- dp_r - \frac{\partial H}{\partial q_r} dt + \sum_1^m a_{rs}\lambda_s dt \right) \delta q_r & \\ + \left(dH - \frac{\partial H}{\partial t} dt + \sum_1^m a_s \lambda_s dt \right) \delta t \Bigg\} &.\end{aligned} \tag{17}$$

The integral of (17) over a Cartan locus can, after an integration by parts, be written as the negative of

$$d \int_{\alpha_0}^{\alpha_1} \sum_1^k p_r \delta q_r - H \delta t - \int_{\alpha_0}^{\alpha_1} \sum_1^m \lambda_s \left(\sum_1^k a_{rs} \delta q_r + a_s \delta t \right) dt, \tag{18}$$

and its integral over an arc of a trajectory is

$$\delta \int_{u_0}^{u_1} \sum_1^k p_r dq_r - H dt + \int_{u_0}^{u_1} \sum_1^m \lambda_s \left(\sum_1^k a_{rs} \delta q_r + a_s \delta t \right) dt. \tag{19}$$

But if we consider a cylinder of trajectories through a Cartan locus and consider the variations from the trajectories as being confined to the surface of such a cylinder, then (18) and (19) may be written

$$d\left\{\int_{\alpha_0}^{\alpha_1}\sum_1^k p_r\delta q_r - H\delta t - \int_{\alpha_0}^{\alpha_1}\int_{t_0}^{t}\sum_1^m \lambda_s\left(\sum_1^k a_{rs}\delta q_r + a_s\delta t\right)dt\right\}, \tag{20}$$

and

$$\delta\left\{\int_{u_0}^{u_1}\sum_1^k p_r dq_r - H dt + \int_{u_0}^{u_1}\int_{\alpha_0}^{\alpha}\sum_1^m \lambda_s\left(\sum_1^k a_{rs}\delta q_r + a_s\delta t\right)dt\right\}, \tag{21}$$

where $\alpha = \alpha(u, v)$, v a parameter, is arbitrary except that

$$\alpha(u, 0) \underset{u}{\equiv} \alpha_0, \qquad \frac{\partial\alpha(u_0, v)}{\partial v} \underset{v}{\equiv} 0, \qquad \frac{\partial\alpha(u_1, v)}{\partial v} \underset{v}{\equiv} 0.$$

The vanishing of (17) identically in δq_r, δp_r, and δt therefore implies, and is implied by, the vanishing of either (20) or (21), for since the cylinder and the law of the Cartan locus given by ρ are arbitrary, the principle of the identical vanishing of coefficients is valid.

Hence, since the λ's are functions of t which are not identically zero, a non-holonomic system which is defined by (5) and (6) of my paper is characterized either by the Cartan integral invariant

$$\int_{\alpha_0}^{\alpha_1}\left\{\sum_1^k p_r\delta q_r - H\delta t - \int_{t_0}^{t}\sum_1^m \lambda_s\left(\sum_1^k a_{rs}\delta q_r + a_s\delta t\right)dt\right\}, \tag{22}$$

or by the Hamilton extremal integral

$$\int_{u_0}^{u_1}\left\{\sum_1^k p_r dq_r - H dt + \int_{\alpha_0}^{\alpha}\sum_1^m \lambda_s\left(\sum_1^k a_{rs}\delta q_r + a_s\delta t\right)dt\right\}. \tag{23}$$

This is Taylor's extension and the corresponding extension of Hamilton's principle for the general case. It might be of interest to note the geometrical significance of each. In the Cartan principle an integral over a Cartan locus is varied by slipping the locus over the cylinder, points going along trajectories, while in the Hamilton principle the same expression is integrated along an arc of a trajectory and variation takes place by keeping the end points fixed but slipping the intervening path over the surface of a cylinder, points going over arcs of Cartan loci.

Finally, two points may be noted.

(1) From the form of (23) Whittaker's remarks which were cited follow as a corollary.

(2) Since (23) does not reduce to (16) as a special case, there are for the restricted conditions to which the original paper applies two different characterizing Hamilton extremal integrals.

THE UNIVERSITY OF ALBERTA

NOTE ON THE CANONICAL FORM OF THE PARAMETRIC EQUATIONS OF A SPACE CURVE BELONGING TO A NON-SPECIAL LINEAR LINE COMPLEX

BY C. R. WYLIE, JR.

In a recent paper,* the author, by means of a projection from hyper-space, obtained the following equations for a general curve belonging to a linear complex,

$$\text{A:} \qquad x_1 = -t, \qquad x_2 = f - \frac{1}{2}tf', \qquad x_3 = -f', \qquad x_4 = 1.$$

It is the purpose of this note to call attention to a more symmetric form of these equations.

Let $x_i = f_i(s)$ be the equation of a general space curve, and $P_{13} = P_{42}$ be the equation of a general linear complex. If the curve belongs to the complex

$$f_1f_3' - f_3f_1' = f_4f_2' - f_2f_4', \text{ or } \frac{f_1^2(f_1f_3' - f_3f_1')}{f_4^2 \cdot f_1^2} = \frac{(f_4f_2' - f_2f_4')}{f_4^2};$$

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* C. R. Wylie, Jr., *Space curves belonging to a non-special linear line complex*, American Journal of Mathematics, vol. 57 (1935), pp. 937–942.