A NOTE ON A PRECEDING PAPER*

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1. *Introduction.* In a paper† by the author, the following lemma was proved.

LEMMA. *If X_s is an element of $A(p)$, the number $\{\sum_{q=1}^{n}(q/n)X_s\vee\}$ is a member of the set $A[1-(1-p)^n]$.*

Then again,‡ the author applied a theorem due to Copeland§. It is our purpose here to extend these two theorems to apply in the field of geometrical probability. The proof of the theorem corresponding to Copeland's follows a different procedure from that given by him. As a matter of fact, the theorem of Copeland may be proved by the method given here.

2. *Extension of the Lemma.* The extension is as follows.

THEOREM 1. *If the numbers $(q/n)x(E_q)$, $(q=1, 2, \cdots, n)$, are such that $x(E_q) = \cdot\phi_{E_q}(P_1), \phi_{E_q}(P_2), \cdots$, where E_q is the interval $0<y\leqq p_q$ and $P_1, P_2, \cdots$ is a set of points admissibly ordered with respect to the function $m(E)$ (the Lebesgue measure of E) defined in Δ; $0<y\leqq 1$, then (1) the number $\sum_{q=1}^{M}(q/n)x(E_q)\vee$ has the probability $[1-\prod_{q=1}^{M}(1-p_q)]$ and (2) the number $\sum_{q=1}^{M}(q/n)x(E_q)\vee$ is a member of the set $A[1-\prod_{q=1}^{M}(1-p_q)]$, where $M\leqq n$.*

PROOF OF (1). We know that‖

$$\text{(a)} \qquad \sim\sum_{q=1}^{M}\left(\frac{q}{n}\right)x(E_q)\vee = \prod_{q=1}^{M}\sim\left(\frac{q}{n}\right)x(E_q)\cdot.$$

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* Presented to the Society, February 29, 1936.

† See the author's memoir *The application of the theory of admissible numbers to time series with constant probability*, Transactions of this Society, vol. 36 (1934), p. 517.

‡ Same reference as above, p. 524.

§ See Copeland, *Admissible numbers in the theory of probability*, American Journal of Mathematics, vol. 50 (1928), p. 550, Theorem 16.

‖ The symbol $\sum(q/n)x(E_q)\vee$ represents the number $\{(1/n)x(E_1)\vee\cdots\vee(M/n)x(E_M)\vee\}$, while $\Pi\sim(q/n)x(E_q)\cdot$ represents $\{\sim(1/n)x(E_1)\cdot\cdots\cdot(M/n)x(E_M)\cdot\}$. Throughout the paper such symbols will have similar mean-

The numbers $\sim(q/n)x(E_q)$, $(q=1, 2, \cdots, M)$, are independent, since $(q/n)x(E_q)$, $(q=1, 2, \cdots, M)$, are independent. From (a), we obtain

$$\sum_{q=1}^{M}\left(\frac{q}{n}\right)x(E_q)\vee = \sim\prod_{q=1}^{M}\sim\left(\frac{q}{n}\right)x(E_q)\cdot,$$

and hence we have

$$p\left[\sum_{q=1}^{M}\left(\frac{q}{n}\right)x(E_q)\vee\right] = 1 - p\left[\prod_{q=1}^{M}\sim\left(\frac{q}{n}\right)x(E_q)\cdot\right],$$

or

$$p\left[\sum_{q=1}^{M}\left(\frac{q}{n}\right)x(E_q)\vee\right] = 1 - \prod_{q=1}^{M}(1-p_q),$$

where $M\leqq n$.

Proof of (2). Since

$$\sum_{q=1}^{M}\left(\frac{q}{n}\right)x(E_q)\vee = \sim\prod_{q=1}^{M}\sim\left(\frac{q}{n}\right)x(E_q)\cdot,$$

then

$$\left(\frac{r_i}{m}\right)\left[\sum_{q=1}^{M}\left(\frac{q}{n}\right)x(E_q)\vee\right] = \sim\prod_{q=1}^{M}\sim\left(\frac{[q+(r_i-1)n]}{mn}\right)x(E_q)\cdot.$$

Then

$$\prod_{i=1}^{k}\left(\frac{r_i}{m}\right)\left[\sum_{q=1}^{M}\left(\frac{q}{n}\right)x(E_q)\vee\right] = \prod_{i=1}^{k}\sim\prod_{q=1}^{M}\sim\left(\frac{[q+(r_i-1)n]}{mn}\right)x(E_q)\cdot.$$

The numbers r_i are chosen such that for every set $r_1, r_2, \cdots, r_k$, we have $0<r_i\leqq m$ and $r_i\neq r_j$ if $i\neq j$. The numbers $\sim([q+(r_i-1)n]/mn)x(E_q)$ are independent. Hence the numbers $\prod_{q=1}^{M}\sim([q+(r_i-1)n]/mn)x(E_q)\cdot$ are independent, from

ings. For the truth of this equality, see Copeland, *The theory of probability from the point of view of admissible numbers*, Annals of Mathematical Statistics, vol. 3 (1932), p. 149.

which it follows that the numbers $\sim\prod_{q=1}^{M}\sim([q+(r_i-1)n]/mn)$ $x(E_q)\cdot$ are independent. We may now conclude that

$$p\left\{\prod_{i=1}^{k}\left(\frac{r_i}{m}\right)\left[\sum_{q=1}^{M}\left(\frac{q}{n}\right)x(E_q)\vee\right]\cdot\right\} = \left\{1 - \prod_{q=1}^{M}(1 - p_q\right\}^{k}.$$

Therefore, the number $\sum_{q=1}^{M}(q/n)x(E_q)$ is an element of $A[(1-\prod_{q=1}^{M}(1-p_q))]$, where $M\leqq n$.

3. *Analog of Copeland's Theorem.* In order to prove the second theorem, we shall need the following lemma.

LEMMA. *If the numbers* $x_1^1, x_2^1, \cdots, x_{N_1}^1, x_1^2, x_2^2, \cdots, x_{N_2}^2, \cdots, x_1^k, x_2^k, \cdots, x_{N_k}^k$ *are such that* $x_j^i\cdot x_{j'}^i=0$, *where* $j\vee j'$ *and* $x_{\gamma_1}^1, x_{\gamma_2}^2, \cdots, x_{\gamma_k}^k$ *are independent, it follows that the numbers* $(x_1^1\vee x_2^1\vee\cdots\vee x_{N_1}^1)$, $(x_1^2\vee x_2^2\vee\cdots\vee x_{N_2}^2), \cdots,$ $(x_1^k\vee x_2^k\vee\cdots\vee x_{N_k}^k)$ *are independent.*

By hypothesis, $x_1^1, x_{\gamma_2}^2, \cdots, x_{\gamma_k}^k$ and $x_2^1, x_{\gamma_2}^2, \cdots, x_{\gamma_k}^k$ are independent, and since $x_1^1\cdot x_2^1=0$, the numbers $x_1^1\vee x_2^1$, $x_{\gamma_2}^2, \cdots, x_{\gamma_k}^k$ are independent.* Then the two sets of numbers $x_1^1\vee x_2^1, x_{\gamma_2}^2, \cdots, x_{\gamma_k}^k$ and $x_3^1, x_{\gamma_2}^2, \cdots, x_{\gamma_k}^k$ are independent, and since $(x_1^1\vee x_2^1)\cdot x_3^1=(x_1^1\cdot x_3^1)\vee(x_2^1\cdot x_3^1)=0$, the numbers $x_1^1\vee x_2^1\vee x_3^1, x_{\gamma_2}^2, \cdots, x_{\gamma_k}^k$ are independent. In general, the numbers $(x_1\vee x_2^1\vee\cdots\vee x_{N_1}^1), x_{\gamma_2}^2, \cdots, x_{\gamma_k}^k$ are independent.

Applying the above to each of the $(k-1)$ remaining groups of numbers, we conclude that the numbers $(x_1\vee x_2\vee\cdots\vee x_{N_1}^1)$, $(x_1^2\vee x_2^2\vee\cdots\vee x_{N_2}^2), \cdots, (x_1^k\vee x_2^k\vee\cdots\vee x_{N_k}^k)$ are independent numbers. Hence we have proved the lemma.

We now come to the analog of the theorem of Copeland.

THEOREM 2. *If the numbers* $(q/n)x(E_q)$, $(q=1, 2, \cdots, n)$, *are such that* $x(E_q)=\phi_{E_q}(P_1), \phi_{E_q}(P_2)\cdots$, *where* E_q *is the interval* $0<y\leqq p_q$ *and* $P_1, P_2, \cdots$ *is admissibly ordered with respect to the function* $m(E)$ *defined in* Δ: $0<y\leqq 1$, *and if*

$$X = \sum_{j=1}^{M} Y_j\vee \text{ and } Y_j = \prod_{i=1}^{\alpha_j}\left(\frac{q_{ij}}{n}\right)x(E_{q_{ij}})\cdot\prod_{i=\alpha_j+1}^{n}\sim\left(\frac{q_{ij}}{n}\right)x(E_{q_{ij}})\cdot,$$

where $0<q_{ij}\leqq n$ *and* $q_{i'j}\neq q_{ij}$ *if* $i'\neq i$, *and where* $Y_{j'}\neq Y_j$ *if* $j'\neq j$, *then* X *belongs to the set* $A(P)$, *where*

* See Copeland, *Admissible numbers in the theory of probability*, American Journal of Mathematics, vol. 50 (1928), p. 543, Theorem 6.

$$P = \sum_{j=1}^{M} \left\{ \prod_{i=1}^{\alpha_j} p_{q_{ij}} \cdot \sum_{i=\alpha_j+1}^{n} (1 - p_{q_{ij}}) \right\}.$$

Since the numbers $(q/n)x(E_q)$ are independent, the numbers $\sim(q/n)x(E_q)$ are independent. By hypothesis, we know that $Y_j \cdot Y_{j'} = 0$ if $j \neq j'$. Hence $p(X) = P$. Now we wish to show that

$$p\left[\prod_{s=1}^{k} \left(\frac{r_s}{m}\right) X \cdot \right] = P^k$$

for every positive integer m and for every set of distinct integers $r_1, r_2, \cdots, r_k$, such that $0 < r_s \leqq m$. We know that

$$\left(\frac{r_s}{m}\right) Y_j = \prod_{i=1}^{\alpha_j} \left(\frac{[q_{ij} + (r_s - 1)n]}{mn} \right) x(E_{q_{ij}})$$
$$\prod_{i=\alpha_j+1}^{n} \sim \left[\frac{[q_{ij} + (r_s - 1)n]}{mn} \right] x(E_{q_{ij}}) \cdot$$

The numbers constituting the above product are independent, and moreover $(r_1/m) Y_{j_1}, (r_2/m) Y_{j_2}, \cdots, (r_k/m) Y_{j_k}$ are independent regardless of whether $j_1, j_2, \cdots, j_k$ are equal or not. Within each group any two distinct numbers are mutually exclusive; that is, $(r_s/m) Y_j \cdot (r_s/m) Y_{j'} = 0$ if $j \neq j'$. We may now apply the above lemma. Hence the numbers

$$\sum_{j=1}^{M} (r_1/m) Y_j \vee, \quad \sum_{j=1}^{M} (r_2/m) Y_j \vee, \quad \cdots, \quad \sum_{j=1}^{M} (r_k/m) Y_j \vee$$

are independent. We know that

$$p\left[\prod_{s=1}^{k} \left(\frac{r_s}{m}\right) X \cdot \right] = p\left[\prod_{s=1}^{k} \left(\frac{r_s}{m}\right) \left\{ \sum_{j=1}^{M} Y_j \vee \right\} \cdot \right]$$
$$= p\left[\prod_{s=1}^{k} \left\{ \sum_{j=1}^{M} \left(\frac{r_s}{m}\right) Y_j \vee \right\} \cdot \right];$$

but since the numbers $\sum_{j=1}^{M} (r_s/m) Y_j \vee$ are independent, the last term is equal to P^k. Therefore the theorem is proved.

It is obvious that the above theorems can be extended to an n-dimensional continuum.

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