INSTRUMENTAL ANALYSIS*

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The use of instruments of computation and analysis is as old as mathematics itself. Counting by the aid of piles of pebbles gave origin to the word calculus. There have always been such aids to reasoning, and the development of instrumental methods has throughout proceeded in parallel with formal methods.

Under instrumental analysis is to be grouped all analysis proceeding by the use of devices for supplementing pure reasoning, whether these devices involve mechanics, optics, heat, electricity or other natural phenomena. The device aids the mind because it approximately obeys some simple law, and may be made to indicate the consequences of combinations of such relationships. An example is the ordinary compass which will draw an approximate circle. One may reason about the properties of families of circles and never draw a diagram; but such reasoning is more surely guided by a few pictures. The instrument, whatever it may be, has two functions: first, when approximate results are sufficient, to yield these directly; second, as a suggestive auxiliary to precise reasoning.

The first piece of mathematical apparatus used was the human hand. Nature, for some reason that is not entirely clear, supplied us with ten fingers, and in the groping development that characterized the beginnings of mathematics, this settled the decimal system upon us. Our forebears apparently did their reckoning standing up, or we might, as in fact did more than one tribe, have used toes as well, and thus landed on the vicenary system. Had this occurred it would have been possible to register all of New York's automobiles with five figures; and Congress would probably have had to be satisfied with merely eight figure appropriations. It is certainly fortunate that at least both hands were used, or the newspapers would now be running out of zeros.

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How Archimedes managed to approximate π as well as he did by an involved geometrical computation in Greek numerals has not been adequately explained. The world suffered long for lack of positional numeration, and for a symbol for zero. Neither of these were invented by formalists. Both were the product of instrumental analysis. They came as the direct and inevitable result of the use of the abacus.

To the occidental it is sometimes necessary to explain that the abacus is the most widely used mechanical computing machine, even today. An expert in its use can nearly keep up with the operator of a simple keyboard-adding-machine, and he can multiply as well. It is a very old device, and no one knows its inventor. Originally, probably, a group of piles of pebbles, it developed into a form in which counters are strung on parallel rods or wires. The mechanical fact that it is convenient to mount rods or wires parallel to one another in a frame produced the idea of positional numeration, and the necessity for noting down complete absence of counters under such circumstances gave us the zero. That the world had to wait so long for these important ideas occurred because the formalists of the time declined to use the plebeian abacus in connection with their profound meditations. So they stuck to their cumbersome notation, while men of trade, with a mechanical aid, produced the most far-reaching of mathematical inventions. The formalists insisted that ruler and compass were the only tools worthy of the gentleman scholar, and by this insistence directed the attention of the learned for centuries to three impossible problems, made so by the artificiality of the limitations. Even today the race of angle trisectors has not died out, although the problem was resolved over a century ago.

Thus, from the earliest times, and in the most profound ways, the use of instruments has influenced the course of formal mathematics. The fixing of the decimal system by the possession of ten fingers, the origination of positional numeration and the zero by the users of the abacus, the cramping of the mathematical style of the Greeks by the ruler and compasses, are incidents in the process. The instrument has been much more than the useful aid; it has often exerted a determining control over the course of events, helpfully or otherwise. This situation is not likely to terminate, and in fact there is every reason to expect that the interrelation will be more complex, if not more intimate, now that mechanical devices may employ a host of new elements and processes. Yet there is a considerable gulf between formal mathematicians and those who make and use mathematical instruments. It is the purpose of this paper to suggest that formal mathematics may again be influenced by instrumental development, and that there would be mutual benefit if the gulf between them were less wide.

When Babbage struggled to put into effect his glorious ideas of mechanical analysis, he was stopped by the expense and delay inherent in his time in the construction of a really complex device. Today much more complex affairs are built at reasonable cost, by reason of mass production of duplicate parts, modern gauging and materials, and modern processes of fabrication. Moreover, and very important, these complex devices are *reliable*; witness the automatic telephone switchboard or the typesetting machine. This development has had a large effect in the field of arithmetical computation. It will have a comparable effect, ultimately, in the fields of other mathematical instruments. There is, nevertheless, a serious barrier to be overcome before this occurs. Reliability comes, in a complicated machine, only when a great deal of study and experiment is devoted to the design of individual parts, which are then fabricated by methods that produce large numbers of precise replicas at low unit cost. The spread between development and production cost may be enormous: thousands of dollars may be spent in perfecting a simple relay or lever which may later be produced for a few cents each, provided hundreds of thousands are made. This barrier militates heavily against the research tool, which is potentially useful in only a few laboratories. Otherwise there would be available a much greater variety of mathematical instruments than at present, and their performance would be more satisfactory. Yet it appears, in spite of this inherent limitation, that we are at the beginning of an important period of development of machines for higher analysis.

The numerical machine is old; Pascal made an excellent adding machine of which we fortunately have records [1],* and

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* Numbers in square brackets refer to the bibliography at the end of this paper.

Leibnitz one capable of performing the four operations of arithmetic [2]. These early devices involved all of the really basic inventions necessary for the successful construction of machines of this class. Leibnitz performed multiplication by repeated addition, shifting columns as the several digits were treated, and this procedure persists in most modern machines. It remained for Bollé to introduce the multiplication table in the form of a three-dimensional cam, and multiplication is thus performed by fewer operations, as many as there are digits in the multiplier [3]. This of course gave his machine greater speed, and facilitated division as well. It is comparatively recently that such machines can be built at moderate cost, and sufficiently reliable and rugged to withstand continuous hard usage. The combination of such machines with punched cards [4], [5] has made arithmetic into an entirely new affair, and it has enabled large banks, insurance companies, and businesses generally to operate in ways that would be fantastic by longhand methods. There is a great deal more arithmetic and better arithmetic in the world than there used to be. This is indicated by the fact that 10,000 tons of cards are used per year, a total of four billion cards or nearly one card per dollar of a famous recent appropriation. The end of the development is not in sight, and much remains to be accomplished. Part of this is due to the fact that the development of such machines is highly expensive, so that they have been produced to meet the needs of large markets, namely, those of the usual business. The usual business doesn't go far beyond rather simple mathematical operations. Yet, now that there is momentum, the large possibilities of the process will undoubtedly be opened up.

The future development of arithmetical devices would appear to lie along the lines of their employment for more complex calculations than at present. One matter that needs attention is an increase in the amount of information that can be placed on a card. The obvious way, that of decreasing the size of holes, has its limitations, although if photography and photo-cell controls were used a great deal could be thus accomplished. A factor of 1000 in the amount of information which may be stored in a given volume appears thus readily attainable. We now have two main types of punched-card apparatus, that operated by making electrical contacts through the holes, and that operated by mechanical rods which pass through the holes to actual controls. These may later be joined by a type in which a light beam passes through and operates a photo-cell. This opens up large vistas, for the light may be pulsating and made to produce a control depending upon its frequency or duration.

There is another way of increasing the information on a card, that of using combinations. This is just beginning to be used. The usual card is divided into columns, commonly 40 or 80 in number. There are 10 positions in a column. One hole only is punched in a column, and by its vertical position indicates a number. Thus only 80 numbers may be carried by a card. If these numbers are typed by typewriter, the space they occupy is less than 1/10 of the area of the card necessary to carry them by the punched scheme. This is not good space economy. To improve it, combinations are used. Two holes, producing a joint effect, may together designate a number and shorten the column. This is readily extended to letters, and thus 80 letters may be placed on a standard card. However this principle of combinations might be carried much farther. Using combinations of two, five positions are sufficient for 10 numbers, thus doubling the card capacity. Carrying the idea to extremes, although with evident difficulty of practical utilization, much more can be attained. There are 800 positions on a card. Combinations of these by two's allow over 300,000 bits of information to be carried. This would allow roughly 10,000 letters of the alphabet, and their sequence, to be carried on a single card, enough to record the information on three typed sheets of usual size. The use of combinations of holes may evidently be much extended.

Great ingenuity has been shown in obtaining desired sequences of operations; but there is still a great deal of carrying cards from one machine to another, and each problem is unique. A group of columns on a card assigned to a particular number, or set of symbols, is called a field. A stack of cards constitutes a two-dimensional array of fields, and each field may contain a number, to a precision determined by the size of field chosen. Apparatus is available for automatically combining any two such numbers, thus recorded on a given card or adjacent cards, in accordance with any of the four numerical operations, and recording the result in a similar manner. The process of bringing the correct numbers into position for the performance of operations, especially for a long sequence of operations, is now the laborious procedure. It involves carrying the cards from one to another of many machines, and waiting each time for the specific operation to be completed. A master control is here conceivable. This process should also be automatic and performed entirely by machine. It should be sufficient, having punched a stack of cards, then to punch a master card, dictating with complete flexibility the operations to be performed, their sequence, the pairs of numbers in the array upon which the operations are to be impressed, and the position in the array where the results are to be joined to the array. A numerical process, however complex, would then be reduced to the recording of the raw data, and the exact specification by similar record of the desired numerical process, all else being relegated, as it should be, to the machine. Such an arrangement would no doubt be soon worked out if there were sufficient commercial demand. This would be a close approach to Babbage's large conception [**4**] as far as arithmetical processes are concerned. It would complete, for arithmetic, the consummation which Leibnitz visualized for all mathematics.

Quite a lot can be accomplished by arithmetic alone, if its operations can be performed rapidly enough, and combined with suitable facility. Witness the difference engine, projected by Babbage, and suffering inevitably from the high cost of precision construction of the time. Feed one of these machines the skeleton of a table, and it will fill in and print the entire table. It has been used for this purpose to good effect. It consists essentially of adding machines coupled together so that a whole sequence of operations is automatically performed one after another, corresponding in this case to the steps involved in higher order interpolation. Complete machines of this sort have been built by Scheutz and others [**6**]. It is time that a numerical machine were built for which the sequence of operations might be varied at will to cover a large field of utility, but just as fully automatic once the sequence is assigned.

There is not the slightest doubt that numerical machines are destined to be further developed and exceedingly useful. Their influence on the users of arithmetic has been powerful. Business does not seem to realize as yet that it needs anything more than arithmetic in its ordinary affairs, and so other devices dealing with branches of mathematics beyond arithmetic have not had the benefit of aggressive commercial development. They are none the less interesting and important, and ultimately their influence may be as great or greater. They will serve, however, the scientist and research man rather than the man of business, at least at first.

No applied mathematics except arithmetic is ever exact, and even arithmetic often merely has that appearance. So when we turn from arithmetic we find devices that are expected to yield approximations to mathematical processes, sufficiently precise for use in applications. There are two main classes, the operators and the equation solvers. No short summary can be at all complete in regard to either, for the literature of the subject is enormous, and each device requires much exposition if it is to be all understood. This is especially true since there is often such a wide gap between the conception of such an intrument and its actual construction and use, so that details are often important. Hence only a few will be selected for illustration in what follows, with full regret that every interesting development cannot be given its due recognition.

Before a mathematical operation can be performed mechanically by an instrument of the first class, the data must appear in the form of a mechanical record. For this purpose two sorts of records are necessary, those representing numbers, and those representing functions. On the form of record depends the means that may be utilized in performing the operation.

Numbers may be recorded in many ways. The position of holes in a card has proved a very fruitful way. The most obvious is by means of a distance between index lines. This leads directly to the combinations of such distances by geometric means, including link motions. The slide rule of Oughtred [7], the alignment charts or nomograms of d'Ocagne, follow almost inevitably. Some of the modern counterparts of these are highly complex and are likely to be genuinely useful only when the occasion for use is sufficiently extensive to submerge the not inconsiderable time necessary for their mastery. The systematization of nomograms, by identification with determinants [7] has been especially helpful in this regard, for it replaces chance invention and sometimes bizarre manipulation to a standardized affair.

Usually the geometrical combination of distances representing numbers is limited to cases where moderate precision is ample. The much coiled slide rules become rather cumbersome. That this is not an inherent limitation is shown however, by Harrison's machine for evaluating differences [8]. This machine was developed for rapidly and precisely recording the enormous array of differences between wave numbers involved in the analysis of spectra. The numbers are represented by distances to holes punched in a long tape which is folded back on itself. A light above makes records through the holes on a sheet of sensitized paper moving slowly below in a direction perpendicular to that of the tape. The position of the dots then gives the desired differences directly. A long loop insures high precision.

Anything measurable may of course be used to represent a number. Thus the deflection of a spring is especially useful when many products are to be obtained. This was employed, for example, in the Michelson-Stratton harmonic analyser [9], which operated with a finite number of ordinates or components, so that it was a device for combining sets of numbers in predetermined fashion.

The representation of a function is most readily made by means of a simple graph. Once represented, the operations of differentiation and integration are evidently fundamental. There have been many ways of performing the latter, but very few of the former, and this is due to the fact that precision in taking the derivative of a function approximately represented by a curve immediately involves all of the difficulty that would be mathematically expected. Moreover many devices projected for this purpose have attempted the impossible feat of finding a derivative by examining one point. Some, such as the optical devices, have depended upon the ability of the eye to detect readily a discontinuity in the derivative of a curve [10]. It would be interesting to examine the extent to which the eye is able to note discontinuities, in fact there seems to be some connection between this matter and the artist's idea of a pleasing curve, but neither the psychologists nor the artists seem to have investigated the subject systematically.

The most reasonable way to obtain a derivative mechanically appears to be to measure the average slope over an adjustable known interval, and this Sears does very neatly [**11**]. The curve is made into an optical mask, which may be of either the variable area or the variable density type, and this is oscillated in the direction of abscissas in front of a slit. Light passing the slit is summed by a photo-electric cell. The fundamental component of the alternating photocell current is then a measure of the derivative. Both the width of slit and amplitude of oscillation are adjustable, to adapt to varying degrees of precision in the curve.

One of the easiest ways to take a derivative is electrically. If the form of a varying current through a pure inductance represents a function, the voltage across the inductance represents the derivative. Unfortunately it is not easy either to cause a current to vary precisely in a prescribed manner, nor to measure precisely a varying voltage. The combination of mask, optical system, and photo-cell will accomplish the current control. This has appeared, combining sinusoids, in acoustical devices for electrical organs, and of course in the ubiquitous talkies. None of these applications require the precision desirable in mathematical instruments. A varying voltage may be very conveniently measured by modern cathode-ray oscillographs, but again not with the precision desirable for purposes of analysis.

The integral of a function represented by a varying current may be obtained by passing the current through a condenser, and measuring the resulting voltage as before. We shall return to this matter of electrical-circuit methods later.

When a function is represented by a graph, the most obvious way to obtain an integral is to cut the curve out and weigh it. This yields a definite integral, and not a running integral, as does in fact the simplest planimeter. Planimeters are noteworthy as instruments which are mechanically simple and mathematically complex. The extreme is the old knife-edge or hatchet planimeter of Prytz [**12**], which works surprisingly well, and causes so much involvement when one is called upon to explain why.

Cutting out a mask to represent a function, uniformly illuminating it, and collecting the light that goes through on a photo cell, gives an integral conveniently, but not highly precisely on account of the limitations of photo-cells and optical systems. It does, however, give a possibility of interesting combinations, which will be mentioned later.

Any adjustable-speed drive may be made into an integrator. If the setting of the adjustment of the ratio of drive is made to vary in accordance with a given function of a variable, while one shaft turns in accordance with the variable itself, the rotation of the output shaft gives the integral. If the shafts are interchanged the output then gives the integral of the reciprocal of the function. James Thomson's disc-ball-and-cylinder integrator was of this type [**13**]. The ball contacts the disc at a point on a diameter, and the cylinder at a point on an element, and transmits motion between them. The disc is turned in accordance with the independent variable, the displacement of the ball from the center is regulated in accordance with the function to be integrated, and the rotation of the cylinder yields the integral. Its modern form, in the hands of Hannibal Ford, has become a rugged and reliable instrument [**14**]. Ford's form has two balls, with direct pressure, and the presence of two balls allows displacement under pure rolling. Maxwell gave a discussion of this type of integrator [**13**], but unfortunately not a complete analysis. If one assumes no backlash, no plastic deformation, and no slippage, the performance is exact; but in a physical instrument the areas of contact are finite. One point only of the area is a point of zero relative motion, and this point shifts in the area when there is any torque, due either to friction or inertia.

With high pressure on the balls considerable load may however be carried and with remarkable precision, although care must be taken in regard to mechanical hysteresis.

For highly precise work the disc and roller type [**15**] appears preferable, for here the area of contact is greatly reduced, and indeterminateness of the position of the zero point is rendered less important. The edge of the roller is hard and fairly sharp, and its shaft is actually in jewel bearings. Such a unit is capable of excellent performance, but it can carry practically no load. Hence, to be useful, it can merely control rotation, the work being derived from another source. The most convenient way is by a torque amplifier [16], which is a device whereby one shaft moves another precisely in step, but with much larger torque delivered than supplied. This works on the same principle as the capstan or windlass, and depends upon wrapped bands. The input shaft pulls one end of the band, and the band rubs on a continuously revolving drum and pulls the output shaft along. Two drums provide for both directions of rotation. Units are in continuous operation giving multiplying factors as high as 10,000.

After simple integration the next important operation is the integration of a function multiplied by a standard kernel. The most prevalent form of device for performing such an operation is the harmonic analyser, for evaluating the integral of the product of a function by a sinusoid. Great ingenuity has been shown in devising such instruments, and scores have been invented. It is not much exaggerated to state that as many forms have been invented as there are actual instruments in present use. Perhaps this is not undesirable, for it is certainly much more pleasant to invent a device of this nature than it is to operate the finished product. The writer pleads guilty to having invented several, none of which are in use. Most harmonic analysers are mechanical combinations of variable speed drives used as integrators, with link motions or cams for introducing the sinusoidal factor. Such, for example, is the Chubb machine [17]. The most convenient and precise is the Henrici-Coradi [18], which introduces its sinusoidal components by means of rollers operating on the surface of glass spheres. There are, however, optical devices, those depending on spring deflections, devices for displacing fluids, and many others. An harmonic analyser, when inverted, becomes a synthesizer. Such are, for example, the tide predicting machines, of which the first was Kelvin's [19].

Of greater mathematical scope, however, is a device for obtaining instantaneously the integral of the product of two given functions. Suggested by Wiener, it has been developed over several years by Gould, Gray, and Hazen [20]. The functions are represented by optical masks placed in parallel planes. The light from a linear light source which passes through both masks is summed by a photo-cell, and gives the measure of the integral of the product. If one of the masks is now shifted in the direction of abscissas, the effect is to shift the origin of one of the functions. The value of the light received plotted against the shift thus evaluates an integral with a cyclic kernel as a function of the parameter under the sign of integration. Work is now under way which aims, in effect, at deforming one of the masks in predetermined fashion, thus evaluating an integral with a general kernel. Such an instrument, which has been called a cinema integraph, is immediately applicable for the evaluation of Fourier transforms, for correlation analysis, and many other purposes.

There is a very fundamental difference between this last instrument and the usual integrator, a difference of profound mathematical importance. Evaluating an integral of a product of functions is one thing, but examining the variation of the integral when one of the functions is deformed or shifted is quite another. There are many devices for performing operations. The most interesting effects occur, however, when these are combined.

The more important class of instruments is therefore the second, the equation solvers. There are three principal types. In the first, simple analogy is used. To solve the equations controlling the performance of a given system, a second system is set up which obeys the same laws, and its performance is measured. The benefit of the substitution is merely one of convenience and precision in measurement. Thus, it is relatively easy to measure the flow of a fluid and hard to measure the flow of heat. Hence, if one wishes to examine the transient flow of heat in a complex device, where analysis is out of the question, such, for example, as a complicated system of hot pipes imbedded in an insulator, one may construct an analogous system where the flow of heat is replaced by the flow of a viscous fluid and measure that. All that is necessary is to provide an apparatus representing the cross section of the system by two plates close together with holes representing the pipes, and supply fluid to each of these holes at pressures proportional to the corresponding pipe temperature. Inserting dye at spaced points will give the flow lines, as was done by Hele-Shaw and his collaborators [**21**]. Collecting and measuring the fluid gives the total heat flow directly.

A variation of this scheme is to employ a system of the same general type, but on a different and usually reduced scale. The dimensions and parameters are so chosen that the original equations still hold, with modification only by multiplication of variables by constants. The systems are, in other words, arranged to be dynamically similar. This is the scheme of the wind tunnel [**22**]. A small airplane, or a small ship, is tested and the results then interpreted to apply to the full-sized device. In each case the velocity through the fluid is usually reduced, as well as the dimensions. There is a complication in this process. The resistance to the passage of a solid through a fluid is of at least two sorts, and they obey different laws. Thus a ship has wave resistance and skin resistance [**23**]. The same scheme of scaling down will not apply to both. Hence it is necessary to render one negligible or measure it separately, and great difficulty in attaining high precision ensues, which is the reason that the use of towing tanks and wind tunnels causes so many arguments, and the reason why large ones are preferred.

In the second type of equation-solver, the constants of the system being studied are independently represented, and this leads to flexibility. An electrical example of this second type of instrument for the solution of equations is given by the Network Analyser [**24**]. It is primarily intended for the examination of complex electrical power networks, and it hence solves certain restricted classes of simultaneous algebraic equations with complex coefficients. It consists merely of a set of condensers, coils, and resistances which may be connected together by plugs. There is thus produced an exact small-scale electrical replica of the system being studied; a miniature network having the same electrical proportions as the actual system. To this are applied, at various points, voltages proportional to those applied to the actual network, and in the same phase relations. The voltages, and hence the currents, are scaled down, and hence the model may be physically small. Measurements made with ordinary instruments yield the performance of the full-size system under interesting operating or emergency conditions. A power system covering several states is thus compacted into a single room for detailed examination. This type of instrument forms an intermediate step between the simple model and the completely flexible equation-solver, for the Network Analyser may be reconnected to represent any desired power system within its limits, and thus one instrument serves for the solution of problems concerning many power networks.

The third type of equation-solver is, however, the one with which we are now most concerned. Like the others, it provides a new system, which obeys the same laws as the one under investigation, and which may be readily and precisely measured, but it does so in a different way. Instead of carrying over an entire equation from a system to its substitute, or even the constants of the equation as in the Network Analyser, the terms of the equation are carried over individually as such and then recombined. This gives great flexibility, for, since the mode of combination is under control, many diverse systems may be studied by a single substitute instrument. As soon as elements are available for performing the operations indicated in an equation, whatever they may be, one thing only is necessary to make an instrument for solving the equation. That is to close the train of operations, or "back-couple" the device. If the equation admits of a discrete solution, this process fixes the parts in position. If the solution is a relation between variables, the device is constrained to move only in accordance with that relationship.

The instrument for solving simultaneous linear algebraic equations gives an example. Wilbur has produced a mechanical device for this purpose, modifying a suggestion which originated, as did so many ingenious ideas, with Kelvin [**25**]. Unknowns are represented by the angles of platens. The positions of a set of pulleys represent the coefficients. Steel bands running over these pulleys have an effective lengthening or shortening representing the terms of the equations, each band corresponding to one equation. Each band passes over a pulley on each platen, thus summing the terms of the corresponding equation. On setting the ends of the bands on indices representing the constant terms, the entire device is fixed in position, and the angles may be read and hence the unknowns determined. The device is entirely geometrical, and no elastic deformation of parts is employed. Before the last band is anchored to its index, all platens are free to move. Fixing this last band completes the set-up and locks every element in position, provided the equations are determinate.

Mallock has built an electrical device for this same purpose [**26**]. The variables are represented by alternating currents in closed circuits, which are coupled together by transformers with adjustable numbers of turns, made proportional to the coefficients, there being as many coils on the transformer as there are variables, plus one to represent the constant terms. Since the total ampere-turns on a transformer must be zero, except for a small excitation, the equity is thus forced. Mallock has a very ingenious way of supplying the excitation separately, to avoid error.

Higher order algebraic equations may be handled in various ways, but so far not precisely [**27**]. The use of the displacement of solids dipped into liquids, as exemplified by a proposal by Meslin, gives one method of attack.

The third type of equation-solver is best exemplified by the differential analyser [**15**], which is an instrument for solving ordinary differential equations. It consists essentially of a set of integrators and means for interconnecting them. Kelvin made the suggestion of this procedure [**28**], but much development was necessary to reduce the idea to practice. There is no limit to the complexity of the equations which may be treated except the number of units available. Several differential-analysers have now been built, and are being used on a large number of problems. Detailed descriptions are available, so only a few points will be mentioned.

The manner of "back-coupling" the device is of especial interest. A single integrator is merely an instrument for performing the operation of integration. If its disc be turned at constant speed, having at any time an angle x, and if its displacement be varied in accordance with a variable y, its output will yield the value of $\int y\,dx$. Suppose now, however, that its output be connected to the y shaft. This is back-coupling. It forces y at all instants to be equal to the output of $\int y\,dx$. The machine is now constrained, and can move only in one definite way, namely, in accordance with the equation $y=\int y\,dx$ or $dy/dx=y$.

It thus yields the exponential solution of this equation, if simultaneous readings be taken of the positions of the x and y shafts. This is the simplest possible sort of back-coupling. Actually, in interesting problems, several integrators are interconnected, together with differential gears for adding, and output tables for introducing variable coefficients, before the final connection is made which fixes the performance of the machine.

The way in which an equation is placed upon the machine may be traced. It is first solved formally for the highest order derivative, and integrated once formally, to yield the derivative of next lower order. A shaft is assigned to this, and one to the independent variable. An integrator may now be connected to yield the derivative of next lower order, and so on until the dependent variable is reached. As we proceed in this manner, every term in the equation becomes represented by the revolutions of a shaft. These shafts are then interconnected through differential gears so that their revolutions sum to zero. This closes the equation, and in effect provides for the drive of the shaft assigned to the next to highest order derivative. When the independent variable shaft is now turned, every other shaft is driven, and the machine is constrained to move in accordance with the equation. The initial conditions are introduced by setting the starting portions of the integrators.

Two terms may be multiplied by a separate mechanism, or better by summing their cross integrals. The square of a variable may be obtained by integrating it against itself. Functional coefficients may be introduced manually as the solution proceeds; but it is more convenient, and also more accurate, to generate them wherever possible. Any coefficient which can be obtained as the solution of an ordinary differential equation may be thus generated, by assigning a portion of the machine to this duty. When thus treated an ordinary differential equation with variable coefficients becomes a more complicated set of equations, but with constant coefficients.

Manual introduction of functions has however one interesting consequence. The procedure is as follows: An index lies on a plotted curve. It is driven in the direction of abscissas by any desired variable by coupling to the machine at the proper point. It is cranked in the direction of ordinates by hand so as to remain on the curve. The crank also turns the shaft in the machine which represents the function. But this makes possible another procedure. The curve being followed may be the actual record being traced by the machine as a solution, the abscissal motion of the index being controlled by a separate variable, a fraction of the variable of abscissas of the curve, or some other. A wide range of functional equations may thus be handled.

It is interesting to speculate on the effect of coupling the cinema integraph to the differential analyser. One of the terms of the equation being solved may then involve a variable parameter under the sign of integration, and the other terms may be those of an ordinary or functional differential equation.

This does not, of course, treat the type of integral equation which is equivalent to a partial differential equation plus a set of boundary conditions. The cinema integraph is adapted to do this by successive approximations, but not directly. The differential analyser also has possibilities in this regard, for it can handle the equations of the characteristics and thus make the solution indirectly. The direct approach would require the continuous deformation of one of the masks in such manner as to cause equality of the output to a specified combination. This has not yet been attempted.

This brings us to the final section of this paper. What may be the mutual influence in the future between machines of the sort considered and formal mathematics?

The machine will not, of course, yield a formal result; it will give only approximate solutions. But its limitations are not at all those usually attending formal solutions. The machine does not care how complex the expression for a coefficient may be, so long as it may be plotted. Discontinuities bother it not at all. Bizarre combinations, such as a function of a derivative appearing as a coefficient, are exactly as readily provided for as is the usual case.

There is no use thus solving an equation unless its solution means something concerning a physical system or unless it suggests the form of a formal solution. In the past we have treated physical problems by means of equations that, allegedly at least, could be solved formally, and the equations have been rather simple in their make-up. Is this because nature prefers the sort of equation that we can handle formally? Or has there been a sifting process whereby the simple equations only have survived?

The equations that come to the differential analyser are "reasonable" equations, in that they admit of solution in a reasonable time of operation of the machine; whereas it is readily possible to construct arbitrarily equations which are most unreasonable in this respect. Why is nature so reasonable, or is the reasonableness ours?

Is there any real use or meaning to some of the bizarre equations that can be solved? Certainly the functional equation crops up, although it is sparingly used in physical problems.

Entirely aside from this speculation, there is one influence which is undoubtedly coming, if equation-solvers become as fully developed and as rapid, reliable, and versatile as the arithmetical machines of commerce. Formal attention will be less directed to the mere solution of equations, in order that they may thus be rendered useful; and will hence be more directed to their formulation and interpretation.

It is to be hoped, as well, that formal attention will be directed to the machines themselves. There are many fascinating problems involved, many of them much too mathematical for those who are engaged in the detailed machine developments. An important query is the question of what sort of machines are really worth developing. This involves the mathematician fully as much as the user of the results.

My mathematical friends exclaim over the ingenuity of the formalist in inventing new methods of construction for use in existence proofs. Perhaps photo-cells might be of service.

We are in an age of complex instruments. Out of it will come devices that will revolutionize the use of mathematics, and will profoundly influence some branches of mathematics itself. This process is now beginning, and it is probable that the next decade will see important advances.

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