MOORE ON GENERAL ANALYSIS—I

General Analysis. Part I. *The Algebra of Matrices.* By Eliakim Hastings Moore with the cooperation of Raymond Walter Barnard. Memoirs of the American Philosophical Society, Philadelphia. Vol. I, 1935.

Eliakim Hastings Moore will always be counted among the heroic figures of American mathematics. His life was devoted to research and to the implanting of respect for research in the minds of others. As head of the department at the University of Chicago he came into contact at one time or another with a large proportion of the present generation of mathematicians in America, and many there are who derived from him an ambition to contribute something to the science. Professor Moore's personality was dominating. He was unusually careful in matters of rigor, very severe with students who were guilty of loose thinking, but quick to forgive and encourage.

Like Weierstrass and Lie, Moore published very little of his later work. This was partly because of the comprehensive nature of his problem, and the interdependence of its parts. It was also due to the extremely high standard which Moore set for himself, and his severe self-criticism. He was continually changing and polishing his work and was never willing to give it his final approval. But now his work is finished, and with the sympathetic and able assistance of Professor Barnard it is given to the world.

Moore's *General Analysis* will be published in four volumes of which this, *The Algebra of Matrices*, is the first. The other volumes will be *The Fundamental Notions of General Analysis*, *Generalized Fourier Series and Modular Spaces*, and *The Characteristic Value Problem in General Analysis*. This first volume contains a preface by G. A. Bliss and an introduction to the entire series by Barnard. In this introduction is a sketch of Moore's first General Analysis theory, and of the historical development of the second (present) theory. The reader is then taken into the author's confidence, the underlying ideas and motives of the General Analysis are explained, its triumphs and temporary failures analyzed, and the changes noted which had to be made to mold it into its final form.

One of the striking features of Moore's work is his extensive use of symbolism. His notation is founded upon that of Peano but it has been greatly modified and extended. To a person not familiar with it, it appears formidable, but actually the system is so consistent that it becomes clear after only a short study. To Moore it was not a system of shorthand, but a medium in which ideas could be expressed with greater clarity and rigor. By its use he was led to finer distinctions than most mathematicians are accustomed to make, and to a better realization of his underlying assumptions. Then, too, the careful statement of a theorem in symbolism often gave him a clue to a possible method of proof.

"*The existence of analogies between central features of various theories implies the existence of a general theory which underlies the particular theories and unifies them with respect to those central features.*" This quotation from Moore's New

Haven Colloquium Lectures (1906) is a declaration of faith which inspired the whole development of his General Analysis. Thus in 1915 he noted that his earlier theory interpreted for the n-dimensional Euclidean manifold appeared to rest upon the presence of a finite properly positive (positive definite) Hermitian matrix. He also noted that the theory of quadratic forms in infinitely many variables rested similarly on the infinite identity matrix. Finally, the work of Hellinger could be shown to depend on a symmetric positive definite matrix of infinite order. In attempting to distill from these diverse situations the essence which they have in common, the second General Analysis theory was formulated, resting upon an algebraic foundation of Hermitian matrices.

This first volume is divided into but three chapters, the first of which is concerned with matrices with elements in a non-commutative field, or quasi-field, or number system of type A, as it is called. The quasi-fields of most importance in the theory are the rational, real, and complex fields and their subfields, and the quasi-field of real quaternions. Functions on a single general range $\mathfrak{P}$ to a second range (usually a number system) are called vectors. Functions on two ranges to a third range are called matrices—thus, dropping Moore's terminology for a moment, a matrix $A=(a_{ij})$ is merely a function of the two variables i and j. Since the composite of the two ranges may be regarded as a single range, every matrix may be regarded as a vector. Conversely, every vector may be regarded as a matrix one of whose ranges is singular. The (open) product of two vectors ξ, η is a matrix whose trace $S\xi\eta$ is their inner product. If κ is a matrix and ξ a vector, then $S\kappa\xi$ is a vector whose components are the inner products of the rows of κ with ξ. Finally, the operation S operating on two matrices gives their ordinary matric product.

No satisfactory definition of determinant being at hand, the usual development of matric algebra is not possible and an alternative approach is made necessary. The matrix δ is the identity for the composition process, and matrices λ_r, λ_l such that $S\kappa\lambda_r=\delta$, $S\lambda_l\kappa=\delta$, are called right and left reciprocals of κ, respectively. A finite matrix is non-singular if it has both types of reciprocal, in which case each is unique and the two are equal. A minor is defined as usual, and the rank of a finite matrix is defined as the order of a non-singular minor of maximum order, the rank of the zero matrix being zero. Thus the important theorems on rank are obtained without using the concept of determinant.

For a quasi-field there are two types of linear dependence, left and right. The rank of a matrix gives the number of columns in every maximal set of right linearly independent columns, or left linearly independent rows. There are two types of systems of linear equations, and for each there is an exact analog of the ordinary theory. Thus the general right-hand solution of a non-homogeneous system can be expressed as the sum of any particular solution and the general solution of the corresponding homogeneous system. The latter is a right linear combination of any $n-r$ right linearly independent solutions, r being the rank of the matrix and n the number of unknowns.

Every matrix is shown to be a product of elementary matrices, or generators of the matric group, as they are termed. Two matrices κ_1, κ_2 are called equivalent if κ_2 (say) can be factored in the sense of composition into a product of three matrices, the middle one being κ_1 and the others non-singular. A matrix of rank r is in normal form if it has zeros everywhere except on the main diagonal, and if this contains r 1's and the other elements are 0. Every matrix can be reduced to normal form by multiplying it on either hand by elementary matrices.

In the second chapter it is further assumed that the quasi-field admits a conjugate process or reciprocal automorphism of period two. That is, corresponding to every number a there is a conjugate $\bar{a}$, whose conjugate is again a, such that the conjugate of a_1+a_2 is $\bar{a}_1+\bar{a}_2$ and the conjugate of a_1a_2 is $\bar{a}_2\bar{a}_1$. For a (commutative) field the correspondence of each number with itself satisfies these conditions. For complex numbers and quaternions the ordinary conjugate process satisfies. Relative to this relation of conjugacy, an Hermitian matrix is defined as a matrix whose transpose is obtained by replacing each element by its conjugate. For Hermitian matrices a definition of determinant is possible. This definition is a little long to be given here with exactness, but it may roughly be illustrated for $n=4$ as follows: Let (234) (1) be a partition of the integers 1, 2, 3, 4. To this partition let correspond the product $(a_{23}a_{34}a_{42}+a_{24}a_{43}a_{32})$ (a_{11}) with a proper sign attached, the subscripts corresponding to all permutations of 234 starting with 2. For $n=7$, to the partition (123) (4567) would correspond a product of two parentheses, the first inclosing two terms, the second six. The determinant $|a_{ij}|$ is the sum of the products corresponding to all partitions. Since each group of terms in parentheses is scalar, the parentheses are commutative with each other, and the determinant is a scalar. If the elements belong to a field, the ordinary determinant is obtained.

The Laplace formulas for the development of a determinant by elements of a row or column are valid. In developing by the elements of a row the cofactors occur on the right, while for the development by the elements of a column, the cofactors appear on the left. The usual criteria hold for determining the rank of an Hermitian matrix from the determinants of the principal minors. Every Hermitian matrix satisfies its characteristic equation. An Hermitian matrix has an adjoint which is Hermitian. Two matrices κ_1 and κ_2 are conjugate if there exists a non-singular matrix ϕ such that $\kappa_2=SS\phi^*\kappa_1\phi$, where ϕ^* is the conjugate-transpose of ϕ. Under such transformations every Hermitian matrix can be reduced to diagonal form. Next Hermitian bilinear functional operators are introduced, and the chapter closes with a consideration of the subspaces of vectors of a given space and the orthogonal complements of these spaces, orthogonality being relative to an Hermitian operator.

In the third and last chapter, an order relation is introduced into the quasi-field with conjugate relation. To this end it is postulated that among the scalars of the quasi-field $\mathfrak{A}$ occur the norms $\bar{a}a$ of all the numbers a of $\mathfrak{A}$, and these norms have the order relations of the positive numbers. It is now possible to define positive and properly positive Hermitian matrices and forms. An Hermitian matrix is positive if each of its finite principal minors has determinant $\geqq 0$, properly positive if each is >0. The Hermitian form $F_\kappa\bar{\mu}\mu$ is positive if it is $\geqq 0$ for every vector μ, properly positive if, further, it is 0 only when $\mu=0$. The fundamental result is obtained that an Hermitian bilinear form is (properly) positive if and only if its corresponding Hermitian matrix is (properly) positive. An analog of Sylvester's law of inertia is proved.

At this point there is introduced the notion of general reciprocal for any finite matrix κ. It may briefly be defined as the matrix λ such that $SS\lambda\kappa=\kappa$, uniqueness being secured by demanding that each column of λ belong to the linear space determined by the columns of κ^*, and the conjugate of each row of λ belong to the linear space of the columns of κ. This general reciprocal reduces to the ordinary reciprocal if κ is non-singular. The general reciprocal of a (positive) Hermitian matrix is again (positive) Hermitian. It is not always true, however, that the reciprocal of a product is the product of the reciprocals in reverse order.

The book closes with a discussion of the Gramian matrix arising from a positive Hermitian bilinear equation, and the problem of orthogonalization with respect to such an operator. Every finite or infinite sequence of vectors can be orthogonalized with respect to the positive Hermitian bilinear operator B, and if B is proper, the orthogonalization is unique.

Professor Barnard and Dr. Coral are to be congratulated upon the excellent presentation. Besides the general introduction, each chapter is preceded by an adequate summary and explanation of its contents. While Moore's notation is preserved throughout, the principal definitions and theorems are explained in words as well as in symbols, and everything possible has been done to make the book comprehensible to the reader. The difficult typography is excellently done.

Finally, it must not be forgotten that Barnard has himself made important contributions to the work which are more than editorial in character, although modesty has forbidden him to point these out.

C. C. MacDuffee

REICHENBACH ON PROBABILITY

Wahrscheinlichkeitslehre. By Hans Reichenbach. Leiden, Sijthoff, 1935. 451 pp.

A complete and perfect work on probability would tell what probability is, would give all mathematical developments in full, and would act as a guide to the application of the theory. The present work by Reichenbach has much of mathematical interest, and deals with a wide range of uses of the probability concept; but its greatest importance comes from the fact that it brings us definitely closer to an understanding of probability itself. This achievement is the more valuable in view of the many conflicting definitions which have been given. Reichenbach himself uses three definitions. The first is purely axiomatic; the second and third, frequency-limit and point-set definitions, are proved to be applications of the first. Other concepts, such as that based on equal probability or the psychological ones, he shows to be either untenable or reducible to the frequency-limit definition.

The author comes to the study of probability from the fields of both physics and philosophy. As a physicist he sees that we can not know that any rigorous laws rule the universe; nor yet—assuming this to be a world of law—the exact formulation of any law. Nor—assuming we knew a precise law to hold—can we measure with precision either the present state on which that law would base a prediction, or the future state which would verify it. So it is that a physicist, even though he ignore the added uncertainties introduced by quantum mechanics, finds the theory of probability essential to an understanding of our relations with nature. "We are dealing here, not alone with problems of