bers $p^{n_1+m}(E_n-M)$ are all finite. Thus we have the following theorem.

THEOREM 5. *Let M be a common boundary of three distinct domains D_k, $(k=1, 2, 3)$, such that D_k is u.l.i-c. for $0 \leqq i \leqq n_k$, and $n_1 \geqq n_2 \geqq n_3$. Then $n_1+n_2 \leqq n-3$, and if there exists $m>0$ such that $n_1+m \leqq n-2$ and $n-(n_1+m)-1 \leqq n_3$, the Betti numbers $p^{n_1+m}(E_n-B)$ and $p^i(B)$, $(0 \leqq i \leqq n_3)$, are all finite.**

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ON THE NORMAL RATIONAL n-IC

BY HELEN SCHLAUCH ADAMS

1. *Notation.* A point α of n-space may be represented by the binary form $(\alpha t)^n=(\alpha_1 t_1+\alpha_2 t_2)^n$ with non-symbolic coefficients $\alpha_0, \cdots, \alpha_n$. If $(\alpha t)^n$ is a perfect nth power $(t_1 t)^n$, α will be the point on C^n of S_n whose parameter is t_1, or briefly the point t_1. Also if $(at)^n$ is a binary form, all points which satisfy the linear apolarity condition $(\alpha a)^n=0$ lie on the $S_{n-1}a$ with coordinates $a_0, \cdots, a_n$. The S_{n-p} $(t_1 t)^p(\beta t)^{n-p}$, with parameters $\beta_0, \cdots, \beta_{n-p}$, is the osculating $(n-p)$-space O_{n-p,t_1} to C^n at t_1.† This notat on is helpful in the development of some of the properties of the normal rational n-ic curve. Many of the analogous properties for the case $n=5$ have been found by other methods by A. L. Hjelmann.‡

2. *The Axes of C^n.* An axis of C^n is a line which lies in $(n-1)$ O_{n-1}'s to C^n. The axes of C^n are given by

$$(\alpha t)^n = (t_1 t)(t_2 t) \cdots (t_{n-1} t)(st),$$

parameters s_0, s_1, the t_i being parameters of points of C^n.

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* Thus, although we have no actual example, it is conceivable that there exists, in E_5, a common boundary M of three domains D_k each of which is u.l.i-c. for $i=0, 1$. If so, $p^2(D_k)$ is infinite for $k=1, 2, 3$; and $p^3(E_5-M)$ is finite.

† Grace and Young, *The Algebra of Invariants*, 1903, Chapter 11.

‡ A. L. Hjelmann, *Sur les courbes gauches rationelles du cinquième ordre*, Annales Academiae Scientiarum Fennicae, (A), vol. 3 (1912–13), No. 11.

Any point $(zt)^n$ of the O_{n-1,t_1} determines n O_{n-1}'s whose t's are roots of $(zt)^n=0$. Those t's which are not t_1 determine the only axis through Z not in O_{n-1,t_1}: $(t_1t)(t_2t)\cdots(t_{n-1}t)(st)$. This axis determines the unique image point Z': $(t_1t)(t_2t)\cdots(t_{n-1}t)(t't)$ in $(t't)(rt)^{n-1}$, parameters $r_0, \cdots, r_{n-1}$.

Then the points of any axis to C^n of S_n and the points of C^n are in one-to-one correspondence, and there are n axes through any point Z of an O_{n-1} only one of which lies outside O_{n-1}. There is a one-to-one correspondence between the axes of C^n not in O_{n-1,t_1} and the points of $O_{n-1,t'}$ which establishes the collineation between pairs of points in two O_{n-1}'s. Similarly, there are ∞^{n-s-1} axes in an O_{n-1,t_1}, and the points of each of the ∞^{n-s} O_{n-s-1,t_1}'s not in O_{n-s,t_1} correspond one-to-one with the points of O_{n-s,t_1}.

3. *Axes in an R_j.* Any R_j meets any S_{2n-2j} in an S_{n-j} whose points correspond, by the collineation of §2, to those of an E_{n-j} in O_{2n-2j}. But S_{n-j} and E_{n-j} together fix a point P of O_{2n-2j}. Then in the two O_{2n-j}'s are determined two points P and P' which are images in the collineation, since the two points are intersections of corresponding S_{n-j}'s. But since such points lie on one axis, P and P' determine an axis which lies in R_j. *Then any R_j contains an axis of C^n, and the variety of axes from an S_j which lies in an O_{n-1} is a V_{j+1}^{j+1}.*

4. *The Osculants to C^n.* The $(i-1)$st osculant to C^n at t_1 is $(t_1t)^{n-i+1}(s_1t)(s_2t)\cdots(s_{i-1}t)$ as t_1 varies.* Then the variety of tangents to it is

$$(t_1t)^{n-i}(s_1t)(s_2t)\cdots(s_{i-1}t)(\alpha t),\ \text{parameters}\ \alpha_0,\ \alpha_1.$$

This variety meets the O_{n-i} $(s_1t)(s_2t)\cdots(s_{i-1}t)(s_it)(\beta t)^{n-i}$, parameters $\beta_0, \cdots, \beta_{n-i}$, in the curve

$$C_{t_1}^{n-i}:\qquad (t_1t)^{n-i}(s_1t)(s_2t)\cdots(s_it)\ \text{as}\ t_1\ \text{varies},$$

which is of the $(n-i)$th order. Obviously the O_s's to $C_{t_1}^{n-i+1}$ form O_{s-1}'s to $C_{t_1}^{n-i}$ and $C_{t_1}^{n-i}$ is the ith osculant of C^n at t_1.

* The first osculant is discussed by G. Castelnuovo in *Studio dell'involuzione generale sulle curve razionali mediante la loro curva normale dello spazio a n dimensioni*, Atti Reale Istituto Veneto, (6), vol. 4 (1885–6), p. 1173; and by St. Jolles, *Die Theorie der Osculanten und des Sehnensystems der Raumcurve* IV *Ordnung* II *Species*, Aachen, 1886.

Thus the ith osculant $C_{t_1}^{n-i}$ is the locus of the points of intersection of the tangents to $C_{t_1}^{n-i+1}$ after the $n-i$ lines common to the variety of the tangents and O_{n-i} are removed. It is normal and rational.

5. *The n-ahedra in an O_{n-1,t_1} Determined by O_{n-1}'s from the Points of r.* Considerany point $Y(yt)^n$ on the line $r(\alpha y+\beta z)^n$ which is not an axis, and does not meet C^n. The n O_{n-1}'s from Y to C^n intersect the O_{n-1,s_1} in the O_{n-2} $(t_1t)(s_1t)(\alpha t)^{n-2}$, parameters $\alpha_0, \cdots, \alpha_{n-2}$, where the t_1's are roots of $(t_1y)^n=0$. Now, the n O_{n-2}'s from Y must osculate the first osculant by §4, and they form an n-ahedron in O_{n-1,s_1}. If $Z(zt)^n$ is a second point of r, the n-ahedron determined by any point of r will have vertices of the type

$$\prod_{j=1}^{n-1}(\alpha t_{1,i+j}+\beta r_{1,i+j})(s_1t), \tag{1}$$

where the t_1's are roots of $(t_1y)^n=0$, and the r_1's are roots of $(r_1z)^n=0$.

Let $\sum a_{ik}x_ix_k'$, $(i,k=0,1,\cdots,n)$, be a quadric V_{n-2}^2 in O_{n-1,s_1}. In order that any two n-ahedra (1) determined by α_1, β_1 and α_2, β_2 be self polar with respect to V_{n-2}^2, $n(n-1)/2+n$ conditions must be satisfied, one more than the number required to determine V_{n-2}^2. Then the two n-ahedra can be self polar with respect to one V_{n-2}^2 only if the determinant Δ of all the expressions $a_{ik}x_ix_k'$ vanishes. But if Δ is arranged so that the $n(n-1)/2$ rows which express the condition that the first n-ahedron be self polar appear first and the n rows relating to the second follow, then Δ can be so reduced that the nth and $(n-1)$st rows involve the same functions of the t_1's, r_1's, α_1, and β_1, while the last two rows involve those same functions of the t_1's, r_1's, α_2, and β_2. The differences of the elements of the two rows will then be constants in each case, and will be the same constants, so that the value of Δ is zero.

Thus the O_{n-1}'s from points of a line which is not an axis and does not meet C^n form n-ahedra in an O_{n-1,s_1}. Obviously, the vertices of all the n-ahedra determined by r form a locus K^{n-1} which is in one-to-one correspondence with $C_{t_1}^1$ and is thus of order $n-1$. Also, there exists a single quadric variety V_{n-2}^2 in O_{n-1,s_1} with re-

spect to which the n-ahedra determined by all points of r are self polar.

6. *Apolarities of Points Related to C^n.* Let $B(bt)^{n-1}$ and $D(dt)^{n-1}$ be two points of O_{n-1,s_1} which are polar with respect to V_{n-2}^2. Then if $(et)^{2n-2}\equiv(bt)^{n-1}(dt)^{n-1}$, the condition that B and D be polar is $(ae)^{2n-2}=0$. But the $(n-1)$ O_{n-2}'s to $C_{t_1}^1$ from B and D have points of contact which are roots of $(t_1b)^{n-1}=0$ and $(t_1d)^{n-1}=0$, respectively. All $2n-2$ of these points are represented by $(et)^{2n-2}=0$. Also the curve $C_{t_1}^1$ meets V_{n-2}^2 in the points which are roots of $(at)^{2n-2}=0$. And since $(ae)^{2n-2}=0$, $(et)^{2n-2}$ and $(at)^{2n-2}$ are apolar.

Let $(t_1^{(i)}t)^n$, $(i=1,\cdots,s)$, be s points of C^n. The O_{n-1}'s at these points meet in an S_{n-s} containing a $C_{t_1}^s$. Now, any of the ∞^{s-1} S_{n-1}'s through S_{n-s} meet C^n in points which are roots of

$$\sum_{i=0}^{s} k_i(t_1^{(i)}t)(t_1t)^{n-1} \equiv (my)^n = 0.$$

Also any point Y of S_{n-s} determines points of osculation of O_{n-1}'s whose parameters are roots of $(y_1t)^n=0$, where s of the t_1's are the $t_1^{(i)}$'s. But since Y lies in all s $O_{n-1,t_1^{(i)}}$'s, $(my)^n=0$.

Then there are the following apolarity relationships among points related to C^n. The binary form representing the points of osculation of the O_{n-2}'s to $C_{t_1}^1$ from any two points of O_{n-1,s_1} which are polar with respect to V_{n-2}^2, and the binary form representing the points of intersection of $C_{t_1}^1$ with V_{n-2}^2 are apolar. Also, the ∞^{s-1} binary forms of the nth order representing points of intersection with C^n of S_{n-1}'s through the S_{n-s} of an sth osculant are apolar to the binary form of the sth order representing the s points of C^n which determine the sth osculant.

7. *$K_{t_1}^{n-1}$ Related to Two Bundles of S_{n-2}'s in O_{n-1,s_1}.* The ∞^{n-2} S_{n-1}'s through r meet O_{n-1,t_1} in ∞^{n-2} S_{n-2}'s forming the bundle (P_{t_1}):

$$(bt)(t_1t)(\delta t)^{n-2}, \text{ parameters } \delta_0,\cdots,\delta_{n-2}.$$

Since if r is not an axis, every O_{n-1} meets r in a point, O_{n-1,t_1} does, and the vertex of (P_{t_1}) is this point. Call it Y_{t_1}. If (P_{r_1}) is a similar bundle in O_{n-1,r_1}, an S_{n-k-1} of (P_{t_1}) will correspond to an S'_{n-k-1} of (P_{r_1}) if they are cut out by the same S_{n-k}. Then corresponding lines of (P_{t_1}) and (P_{r_1}) are

$$(t_1t)(\alpha t)^{n-2}(\delta t), \text{ parameters } \delta_0, \delta_1, \alpha\text{'s fixed},$$

and

$$(r_1t)(\alpha t)^{n-2}(\delta t), \text{ parameters } \delta_0, \delta_1, \alpha\text{'s fixed}.$$

By the projectivity of §2, Y_{t_1} is a point of $K_{t_1}^{n-1}$. If a line of O_{n-1,t_1} is defined by S_{n-2}'s which contain an axis, it meets its image in O_{n-1,r_1} in the point of $K_{t_1}^{n-1}$:

$$(s_1^{(1)}t)(s_1^{(2)}t) \cdots (s_1^{(n-2)}t)(t_1t)(r_1t).$$

Then if r is not an axis, $K_{t_1}^{n-1}$ is the locus of points of intersection of corresponding lines of (P_{t_1}) and (P_{r_1}) when the S_{n-2}'s defining the line of (P_{t_1}) contain a fundamental axis, and the vertex of (P_{t_1}) is a point of $K_{t_1}^{n-1}$. If r is an axis, it meets O_{n-1,t_1} in a point Y of $K_{t_1}^{n-1}$ which corresponds in (P_{r_1}) to Y_{r_1}, the image of Y_{t_1} of (P_{t_1}). Thus if r is an axis, all $K_{t_1}^{n-1}$'s of O_{n-1,t_1}'s not defining r are the intersections of corresponding lines of two bundles (P_{t_1}) and (P_{r_1}).

By the correspondence between S_j's of (P_{t_1}) and (P_{r_1}), it can be shown that *the bisecant lines, $\cdots$, the j-secant S_{j-1}'s, $\cdots$, the $(n-2)$-secant S_{n-3}'s of $K_{t_1}^{n-1}$ are formed by the intersections of corresponding S_2's, $\cdots$, S_j's, $\cdots$, S_{n-2}'s of (P_{t_1}) and (P_{r_1}).*

8. *The Principal $(n-2)$-ic of S_{n-i} Associated with C^n of S_n.* Let C'^n be the image curve of C^n projected upon any S_{n-i} from a vertex S_{i-1} not containing points of C^n. It is obvious that this image curve is of the nth order and that it is in one-to-one correspondence with C^n. Obviously the image in S_{n-i} of any line meeting the vertex S_{i-1} is a point.

By §2, the variety of axes to C^n is easily seen to be of order $2n-2$. Likewise, the order of the variety of axes meeting a line r of S_{i-1} which is not itself an axis, is $2n-2$. Now from any point Y of r can be passed n O_{n-1}'s to C^n which determine, $n-1$ by $n-1$, n axes through Y, and the dimensionality of the variety of axes through r is therefore 2. *Then the axes meeting r form a surface V_2^{2n-2}.* Of this surface, r is an n-fold directrix since every point of it is n-fold on the surface. Now any S_{n-1} through S_{i-1} also passes through r, which counts for n in its intersection with V_2^{2n-2}. The residual intersection of S_{n-1} with V_2^{2n-2} is then $n-2$ lines of V_2^{2n-2}; or every S_{n-1} through S_{i-1} contains $n-2$ lines of V_2^{2n-2}. It has been shown that the image of an axis through r

is a point; obviously the image in S_{n-i} of an S_{n-1} through the vertex S_{i-1} is an S_{n-i-1}. Since there are ∞^{n-i} positions of S_{n-1}'s through S_{i-1} and ∞^{n-i} S_{n-i-1}'s in S_{n-i},* then every S_{n-i-1} of S_{n-i} contains $n-2$ points of the locus of images of axes through r. Since there are ∞^1 such images, *the locus of the images in* S_{n-i} *of axes through any line* r *of the vertex of projection not an axis is a curve of the* $(n-2)$*nd order called a principal* $(n-2)$*-ic of* S_{n-i} *associated with* C^n. There are ∞^{2i-4} such principal $(n-2)$-ics, one for each position of r in S_{i-1}. It can be shown that *if* r *is an axis, the principal curve in* S_{n-i} *is an* $(n-3)$*-ic.*

9. *Projection of the* $K_{t_1}^{n-1}$*'s of* O_{n-1,t_1}. We see that n of the O_{n-2}'s of §4 from a point Y of r (not an axis) intersect $n-1$ by $n-1$ in vertices of an n-ahedron since each is an S_{n-1}. Thus the vertices of such an n-ahedron are images of O_{n-1,t_1} of axes to C^n which meet r. It was also shown in §4, that these vertices lie on a $K_{t_1}^{n-1}$ of O_{n-1,t_1}. Then an O_{n-1} meets the V_2^{2n-2} of axes from r in $K_{t_1}^{n-1}$, and V_2^{2n-2} has upon it ∞^1 curves, $K_{t_1}^{n-1}$. Finally, *the* ∞^1 $K_{t_1}^{n-1}$*'s of the* O_{n-1}*'s project into the principal* $(n-2)$*-ic of* S_{n-i}.

10. *Projection of the n-ahedra of* O_{n-2} *to* C^{n-1}, *n Odd.* It is easily seen that there is an nth order involutorial relation between the points Y and the vertices of the n-ahedra mentioned above. Then from §9 it follows that there is an nth order involution between the points of r and those of any $K_{t_1}^{n-1}$, and that there is an involution of order n defined by the points of r and those of its principal $(n-2)$-ic in S_{n-i}, (n odd). Since the O_{n-1}'s which intersect O_{n-1,t_1} in faces of an n-ahedron discussed here meet r, the resulting O_{n-2}'s determine, with r, S_{n-1}'s which meet S_{n-i} in S_{n-i-1}'s. *Then the projection in* S_{n-i} *of the faces of the n-ahedra are* S_{n-i-1}*'s which, since the original sides were* O_{n-2}*'s to* $C_{t_i}^{n-1}$, *have* $(n-2)$*-fold contact with* $C_{t_i}'^{n-1}$, *and are inscribed in the principal* $(n-2)$*-ic at n points of the fundamental involution*† *on it, n odd.* (*When* $i=2$, *the* S_{n-3}*'s are stationary.*)

* For the proof that S_n contains $\infty^{(n-m)(m+1)}$ S_m's, $(m<n)$, see G. Veronese, *La superficie omaloide normale a due dimensioni e del quatro ordine dello spazio a cinque dimensioni e le sue projezioni nel piano e nello spazio ordinario*, Memorie dell'Accademia dei Lincei, (3), vol. 19 (1884), p. 347.

† For the definition and some discussion of the fundamental involution see L. Berzolari, *Sulle curve razionali di uno spazio lineare ad un numero qualunque di dimensioni*, Annali di Matematica, (2), vol. 21 (1893), pp. 1–25; A. Brill,

11. *The Projection of an Apolarity of* §6, *n Odd.* For the s points of C^n mentioned in §6 may be chosen s points of one group of the fundamental involution. In this case, the S_{n-s} containing the sth osculant passes through a point Y of r and contains s axes. By the projection into S_{n-i}, these s axes determine points of one group of the fundamental involution of the principal $(n-2)$-ic from r. Since S_{n-s} meets r, it projects by the usual method into an $S_{n-s-i+1}$ and the S_{n-1}'s through S_{n-2} project into S_{n-i-1}'s in S_{n-i}. *Then in S_{n-i}, the sth order binary form representing s points of one group of the fundamental involution on the principal $(n-2)$-ic from r is apolar to the ∞^{s-i} nth order binary forms, each representing the points of C'^n lying in S_{n-i-1}'s through the $S_{n-s-i+1}$ containing the image of any sth osculant of C^n, n odd.*

12. *Projected $(n-1)$-ahedra in S_{n-i} Associated with the Points of $K_{t_1}^{n-1}$ in an S_{n-2} of O_{n-1,t_1}.* Since $K_{t_1}^{n-1}$ is of the $(n-1)$st order, its points are given by $(s_1t)^{n-1}(dt)\equiv(st)^n$, d fixed. Now any S_{n-2} $(ct)(\alpha t)^{n-2}\equiv(et)^n$, parameters $\alpha_0, \cdots, \alpha_{n-2}$, will meet $K_{t_1}^{n-1}$ in points whose parameters are the roots of $(se)^n=0$, of which there are obviously $n-1$. Then also an *S_{n-2} in an O_{n-1} will meet $K_{t_1}^{n-1}$ in $n-1$ points.*

Now by the fundamental projectivity, every point of S_{n-2,t_1} is the image of an axis (§2) and since S_{n-2,t_1} contains ∞^{n-2} points, these axes form an $n-1$ dimensional variety. This variety is also composed of the ∞^1 S_{n-2}'s which are homologous to S_{n-2,t_1} in the other O_{n-1}'s; that is, which are cut out of the ∞^1 O_{n-1}'s by the axes of the variety. Among the axes are those which meet S_{n-2,t_1} in the $n-1$ points of $K_{t_1}^{n-1}$ and which must meet every other O_{n-1} in $n-1$ points (in accordance with §5, r is not an axis). These $n-1$ points where the axes from the points in S_{n-2,t_1} of $K_{t_1}^{n-1}$ meet an O_{n-1} determine that O_{n-1}. If, however, the O_{n-1,s_i} is one of those defining an axis to one of these points, then the axis lies completely in O_{n-1,s_i}. In this case, the S_{n-2} homologous to that of O_{n-1,t_1} is defined by the $n-2$ points where the other special axes meet O_{n-1,s_i} and the point of r from which O_{n-1,s_i} was drawn.

Ueber binäre Formen und die Gleichung sechsten Grades, Mathematische Annalen, vol. 20 (1882), pp. 330–357; W. Stahl, *Ueber Fundamentalinvolutionen auf rationalen Curven*, Journal für Mathematik, vol. 104 (1889), pp. 38–61.

Since each S_{n-2,s_i} passes through a point of r which lies in the vertex S_{i-1}, $S_{n-2,j}$ will project into an S_{n-i-1} of S_{n-i}. Also every S_{n-2,s_i} contains a point of the axis which connects a point of r to a point of $K_{t_i}^{n-1}$. Now this whole axis determines a point of $K_{t_1}^{n-1}$ and, since it meets r, must project into a point of the principal $(n-2)$-ic of S_{n-i} associated with r. However, the point common to this axis and S_{n-2,s_i} is on r and thus projects into no portion of S_{n-i}. Thus the images of the other $(n-2)$ S_{n-2,s_i}'s in S_{n-i} contain the same point of the principal $(n-2)$-ic of S_{n-i} associated with r. Then, finally, *the* ∞^{n-1} S_{n-2,s_i}*'s associated with the* ∞^{n-1} S_{n-2}*'s of any* O_{n-1,t_1} *project into* S_{n-i} *in* $(n-1)$-*ahedra whose faces are* S_{n-i-1}*'s, every* $n-2$ *of which meet in a point of the principal* $(n-2)$-*ic of* S_{n-i} *associated with* r.

13. *Projection of the Variety of Axes from Points of a Line of* O_{n-1,t_1}. This V_2^2 was defined in §3. Obviously, *the lines of the variety project into lines which envelop a conic.* If the line of O_{n-1,t_1} is a bisecant of $K_{t_1}^{n-1}$, the conic has two points in common with the principal $(n-2)$-ic.

14. *On the Image of an Axis.* The points determining an axis define, $n-2$ at a time, the $(n-2)$nd osculants $C_{t_1,t_2,\cdots,t_{n-2}}^2$ to each of which the axis is a tangent (§4). Now every axis not meeting the vertex of projection, S_{i-1}, determines with it an S_{i+1} which meets S_{n-1} in a straight line, the image of the axis. *Then the image of an axis, determined by* $n-1$ *points of* C^n, *is tangent to the* $n-1$ *images of the* $n-1$ *quadratic osculants determined by these points* $n-2$ *at a time.*

Since the S_{i+1} determined by the image of the axis and S_{i-1} contains an axis (§3), *any line of* S_{n-i} *may be regarded as the projection of an axis provided* $i>(n-2)/2$. The axis of which any line in S_{n-i} is the image corresponds to $n-1$ points of C^n, which project into $n-1$ points of C'^n, so that *any line of* S_{n-1} *corresponds to* $n-1$ *points of* C'^n.

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