A CHARACTERIZATION OF MANIFOLD BOUNDARIES IN E_n DEPENDENT ONLY ON LOWER DIMENSIONAL CONNECTIVITIES OF THE COMPLEMENT*

BY R. L. WILDER

In my recent paper *Generalized closed manifolds in n-space*† it was shown‡ that a compact point set B in E_n, common boundary of (at least) two domains D and D_1 which are respectively u.l.i-c.§ for $0 \leqq i \leqq j$ and $0 \leqq i \leqq n-j-3$ (where $n-2 > j \geqq (n-3)/2$), and such that the Betti numbers $p^{j+1}(D)$, $p^{j+2}(D), \cdots, p^{n-2}(D)$ are finite, is a g.c.$(n-1)$-m. This constituted a generalization of a former result|| to the effect that when $n=3$, D and D_1 are u.l.0-c., and $p^1(D)$ is finite, B is a closed 2-manifold. In the present note I propose to show, as principal result, that the above conditions on the numbers $p^{j+2}(D), \cdots, p^{n-2}(D)$ are irrelevant, and furthermore that it is immaterial whether we place the restriction as to finiteness on $p^{j+1}(D)$ or on $p^{n-j-2}(D_1)$. It turns out that the only essential requirements are that the upper limits on the dimensions for which D and D_1 are u.l.i-c. must total at least $n-3$, and that one of the domains have a finite Betti number as just stated.

For the sake of brevity we make the following definitions. We shall understand without explicit statement hereafter that the imbedding space is $E_n (n \geqq 3)$ (euclidean space of n dimensions).

DEFINITION. A metric space will be said to be *completely i-avoidable*¶ at a point P if for every $\epsilon > 0$ there exist δ and η, $\epsilon > \delta > \eta > 0$, such that if γ^i is a cycle on $F(P, \delta)$, then $\gamma^i \sim 0$ on $S(P, \epsilon) - S(P, \eta)$.

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* Presented to the Society, December 29, 1934.

† Annals of Mathematics, vol. 35 (1934), pp. 876–903; to be referred to hereafter as G.C.M.

‡ Principal Theorem E of G.C.M.

§ u.l.i-c. = uniformly locally i-connected; see G.C.M. for definition.

|| R. L. Wilder, *On the properties of domains and their boundaries in* E_n, Mathematische Annalen, vol. 109 (1933), pp. 273–306, Theorem 20; to be referred to hereafter as D.B.

¶ See condition (3), definition M^n, of G.C.M.

THEOREM 1. *Let M be the boundary of a u.l.i-c. domain D, $(0 \leqq i \leqq n-j-2)$, and P a point of M at which M is completely $(n-j-2)$-avoidable. Then there exists for every $\epsilon > 0$ an $\eta > 0$ such that if $\gamma^i \subset S(P, \eta)$ links M, then γ^i is linked with a cycle Γ^{n-j-1} of $D \cdot S(P, \epsilon)$, and with a cycle Γ_1^{n-j-1} of $M \cdot S(P, \epsilon)$.**

PROOF. Let ϵ' be an arbitrary positive number $< \epsilon$, and let δ and η be such that a γ^{n-j-2} of $M \cdot F(P, \delta)$ is homologous to zero on $M \cdot [S(P, \epsilon') - S(P, \eta)]$. Suppose $\gamma^i \subset S(P, \eta)$ links M. Let $H = M \cdot \overline{S(P, \delta)}$ and $K = M \cdot \overline{S(P, \epsilon')}$. Then γ^i links K. For suppose not. Then there exists a chain $C_1^{i+1} \rightarrow \gamma^i$ in $E_n - K$, and hence in $E_n - H$. A chain $C_2^{i+1} \rightarrow \gamma^i$ in $S(P, \eta)$ lies also in $E_n - (\overline{M - H})$. Then the cycle $C_1^{i+1} - C_2^{i+1}$ must link $H \cdot \overline{M - H}$, else by the Alexander Addition Theorem γ^i does not link $H + \overline{M - H} = M$. But then $C_1^{i+1} - C_2^{i+1}$ is linked with an $(n-j-2)$-cycle of $M \cdot F(P, \delta)$, since $H \cdot \overline{M - H} \subset M \cdot F(P, \delta)$. But such a cycle bounds on $M \cdot [S(P, \epsilon') - S(P, \eta)] \subset E_n - |C_1^{i+1} - C_2^{i+1}|$,† and we have a contradiction. Thus γ^i links K.

As γ^i links K, it is linked with a cycle Γ_1^{n-j-1} of K. Since D is u.l.i-c. for $0 \leqq i \leqq n-j-2$, there lies in $D \cdot S(P, \epsilon)$‡ a cycle Γ^{n-j-1} approximating Γ_1^{n-j-1} and linked with γ^i.

THEOREM 2. *Let the compact point set M be the common boundary of (at least) two domains D_1 and D_2 such that D_k, $(k=1, 2)$, is u.l.i-c. for $0 \leqq i \leqq n_k$, where $n_1 + n_2 = n - 3$. Also, let (n_1+1)-cycles of D_1 of diameter less than some fixed positive number θ bound in D_1. Then M is a g.c. $(n-1)$-m.*§

PROOF. CASE 1. Suppose $n_1 \geqq n_2$. By Theorem 3 of G.C.M., M is completely i-avoidable at all points, for $0 \leqq i \leqq n_2$. We first prove that D_1 is u.l.(n_1+1)-c. If D_1 is not u.l.(n_1+1)-c., there exist $P \subset M$ and $\epsilon > 0$ such that for each $\eta > 0$ there exists a

* Theorem 1 is a generalization of Theorem 4 of my paper *Concerning a problem of K. Borsuk*, Fundamenta Mathematica, vol. 21 (1933), pp. 156–167. It should be noted that the neighborhoods $S(P, \epsilon)$ are relative to E_n, not merely to M.

† If L is a chain, by $|L|$ we denote the set of points on L.

‡ See Lemma 2a of G.C.M. (A typographical error occurs in the statement of the lemma; the last "j" should be "1".)

§ Theorem 2 is an exact generalization of Theorem 8 of the paper in Fundamenta Mathematica, vol. 21, cited above.

cycle ${}^{\eta}\gamma^{n_1+1}$ in $D_1 \cdot S(P, \eta)$ that does not bound in $D_1 \cdot S(P, \epsilon)$. However, let us choose δ and η to satisfy the complete i-avoidability requirement with $\eta < \theta$. By hypothesis, there exists $K_1^{n_1+2} \rightarrow {}^{\eta}\gamma^{n_1+1}$, in D_1 and hence (for $i = n_2$) in $E_n - H$ (H as defined in proof of Theorem 1). Any $K_2^{n_1+2} \rightarrow {}^{\eta}\gamma^{n_1+1}$ in $S(P, \eta)$ also lies in $E_n - [F(P, \epsilon) + \overline{M - H}]$. Then $K_1^{n_1+2} - K_2^{n_1+2}$ must link a cycle of $M \cdot F(P, \delta)$, else ${}^{\eta}\gamma^{n_1+1}$ bounds in $D_1 \cdot S(P, \epsilon)$. But then it is linked with a Γ^m of $M \cdot F(P, \delta)$, where $M = n - (n_1 + 2) - 1 = n_2$; such a cycle, however, bounds on $M \cdot [S(P, \epsilon) - S(P, \eta)]$, hence in $E_n - (K_1^{n_1+2} - K_2^{n_1+2})$. Thus the supposition that ${}^{\eta}\gamma^{n_1+1}$ does not bound in $D_1 \cdot S(P, \epsilon)$ leads to a contradiction.

We may now show that D_1 is u.l.i-c. for $n_1 + 2 \leqq i \leqq n - 2$. Let j be such a fixed value of i; we note that $n_2 \geqq n - j - 1 \geqq 1$. Suppose D_1 not u.l.j-c. Then we may determine a point P of M and an $\epsilon > 0$ such that for each $\eta > 0$ there is a cycle ${}^{\eta}\gamma^j$ of $D_1 \cdot S(P, \eta)$ that fails to bound in $D_1 \cdot S(P, \epsilon)$. Let δ and η be such that (1) $\epsilon > \delta > \eta > 0$, (2) any $(n - j - 2)$-cycle of $M \cdot S(P, \delta)$ bounds in $M \cdot [S(P, \epsilon) - S(P, \eta)]$, (3) any $(n - j - 1)$-cycle of $D_2 \cdot S(P, \delta)$ bounds in $D_2 \cdot S(P, \epsilon)$ and hence in D_2, and (4) if an ${}^{\eta}\gamma^j$ links M, then (Theorem 1) it is linked with an $(n - j - 1)$-cycle of $D_2 \cdot S(P, \delta)$. Now if an ${}^{\eta}\gamma^j$ of D_1 were linked with M, we could by condition (4) determine an $(n - j - 1)$-cycle of $D_2 \cdot S(P, \delta)$ with which ${}^{\eta}\gamma^j$ is linked. As this would not be possible by condition (3), we can suppose that ${}^{\eta}\gamma^j$ does not link M. Then there exists a chain $K_1^{j+1} \rightarrow {}^{\eta}\gamma^j$ in $E_n - M$, hence in $E_n - M \cdot \overline{S(P, \delta)}$. Let K_2^{j+1} be an arbitrary chain of $S(P, \eta)$ bounded by ${}^{\eta}\gamma^j$, and we have $K_2^{j+1} \rightarrow {}^{\eta}\gamma^j$ in $E_n - [F(P, \epsilon) + M - M \cdot S(P, \delta)]$. As before, we see by applying condition (2) that ${}^{\eta}\gamma^j$ bounds in $D_1 \cdot S(P, \epsilon)$.

Thus D_1 is u.l.i-c. for $0 \leqq i \leqq n - 2$, and for this case the theorem follows from Principal Theorem C of G.C.M.*

CASE 2. Suppose $n_1 < n_2$. In this case we show that D_2 is u.l.i-c. for $n_2 + 1 \leqq i \leqq n - 2$. We note that M is completely $(n - j - 2)$-avoidable for $0 \leqq n - j - 2 \leqq n_1$ at all points. The proof then follows the general method of Case 1.

The following corollary is obvious.

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* That D_1 is simply $(n - 1)$-connected follows from the fact that M, being a common boundary of two domains, is a continuum.

COROLLARY. *A compact set that is the common boundary of (at least) two domains D_1 and D_2 such that D_k, $(k=1, 2)$, is u.l.i-c. for $0 \leqq i \leqq n_k$, where $n_1+n_2=n-2$, is a g.c.$(n-1)$-m.*

THEOREM 3. *Let M be a common boundary of (at least) two domains D_1 and D_2 such that D_k, $(k=1, 2)$, is u.l.i-c. for $0 \leqq i \leqq n_k$, where $n_1+n_2=n-3$. Then if $p^{n_k+1}(D_k)$ is finite for either $k=1$ or 2, there exists a $\theta > 0$ such that (n_k+1)-cycles of D_k of diameter $< \theta$ bound in D_k.**

PROOF. Take, for instance, $p^{n_1+1}(D_1)$ finite. Let $n_1+1=k$. Denote the cycles of a k-basis of D_1 by $\Gamma_i^k, (i=1, 2, \cdots, m)$. By the method of proof of Theorem 5 of G.C.M. we can prove the following lemma.

LEMMA. *Let D be a u.l.i-c. domain, $(0 \leqq i \leqq j)$, and let Γ_i^k, $(i=1, 2, \cdots, m; 0 \leqq n-k-1 \leqq j+1)$, be a set of independent cycles linking $\overline{D}$. Then in D there exist independent cycles γ_i^{n-k-1}, $(i=1, 2, \cdots, m)$, such that every linear combination of the Γ's is linked with at least one γ.*

Applying the lemma, we see that there exists in D_2 a set of $(n-k-1)$-cycles γ_i^{n-k-1}, $(i=1, 2, \cdots, m)$, such that every linear combination of the Γ's is linked with at least one of the γ_i^{n-k-1}. The remainder of the proof is similar to that for Theorem 14 of D.B. From Theorems 2 and 3 we have our principal result.

PRINCIPAL THEOREM. *Let a compact point set M be a common boundary of (at least) two domains D_1 and D_2 such that D_k, $(k=1, 2)$, is u.l.i-c. for $0 \leqq i \leqq n_k$, where $n_1+n_2=n-3$. Then, if one of the numbers $p^{n_k+1}(D_k)$ is finite, M is a g.c.$(n-1)$-m.*

For the case $n=3$, where necessarily the numbers n_1 and n_2 as defined above must equal 0, I have shown in D.B. that without the single condition as to the finiteness of one of the numbers $p^{n_k+1}(D_k)$, not only may the boundary fail to be a manifold, but it may be the common boundary of three or more domains. However, if M has a single point P such that all 1-cycles of $D_k \cdot S(P, \epsilon)$ bound in D_k, then M is the common boundary

* Compare Theorem 14 of D.B.

of only two domains. (Theorem 11 of D.B.) We now extend the latter result to higher dimensions.*

THEOREM 4. *Let M be a common boundary of two domains D_1 and D_2 such that D_k, $(k=1, 2)$, is u.l.i-c. for $0 \leqq i \leqq n_k$, where $n_1+n_2=n-3$. Then, if for (at least) one of the values of k, there exists a point P of M and an $\epsilon>0$ such that all (n_k+1)-cycles of $D_k \cdot S(P, \epsilon)$ bound in D_k, it follows that M is the common boundary of only two domains. Indeed, at P, M is locally a g.c.$(n-1)$-m.*†

PROOF. Let $n_1 \geqq n_2$. As both D_1 and D_2 are u.l.0-c., M is a Jordan (or Peano) continuum, and the component C of $M \cdot S(P, \epsilon)$ determined by P is an open subset of M. By the method of argument used for Theorem 9 of D.B., C is the common boundary of two u.l.i-c., $(0 \leqq i \leqq n_k)$, domains D_k', $(k=1, 2)$, in $S(P, \epsilon)$, where all points of D_k' in a certain neighborhood U (rel. E_n) of C belong to D_k and conversely. As in Theorem 3 of G.C.M. we show that C is completely i-avoidable at all points for $0 \leqq i \leqq n_2$.

We may now proceed, as in Theorem 2 above, to show that one of the domains D_k' is u.l.i-c. for $0 \leqq i \leqq n-2$ at all points of C. Following this, we may show by methods such as those used to prove Theorem 12 of G.C.M. that in U there exist only points of $C+D_1+D_2$.

In conclusion we note that in higher dimensions there exist, a priori, further possibilities concerning common boundaries of several domains. For instance, does there exist for some E_n a common boundary of three domains D_k, $(k=1, 2, 3)$, such that D_k is u.l.i-c. for $0 \leqq i \leqq n_k$, where $n_1 > n_2 > n_3$? The answer, in case $n_1+n_3 \geqq n-3$, is clearly negative by virtue of the corollary to Theorem 2 above; and indeed we must have $n_1+n_2 \leqq n-3$ in such a case. For the case $n_1+n_2=n-3$, let us consider the Betti numbers $p^{n_1+m}(E_n-M)$, where $n_1+m \leqq n-2$ and $n-(n_1+m)-1 \leqq n_3$ (if any such exist). By the proof of Theorem 4 of G.C.M. we may show $p^i(B)$ finite for $0 \leqq i \leqq n_3$. Consequently the num-

* It will be noted that we show now that the "ϵ-condition" is needed only for one domain.

† That is, conditions (2), (3) of definition M^{n-1} of G.C.M. are satisfied for some connected open neighborhood U of P, and so on.

bers $p^{n_1+m}(E_n-M)$ are all finite. Thus we have the following theorem.

THEOREM 5. *Let M be a common boundary of three distinct domains D_k, $(k=1, 2, 3)$, such that D_k is u.l.i-c. for $0\leqq i\leqq n_k$, and $n_1\geqq n_2\geqq n_3$. Then $n_1+n_2\leqq n-3$, and if there exists $m>0$ such that $n_1+m\leqq n-2$ and $n-(n_1+m)-1\leqq n_3$, the Betti numbers $p^{n_1+m}(E_n-B)$ and $p^i(B)$, $(0\leqq i\leqq n_3)$, are all finite.**

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ON THE NORMAL RATIONAL n-IC

BY HELEN SCHLAUCH ADAMS

1. *Notation.* A point α of n-space may be represented by the binary form $(\alpha t)^n=(\alpha_1t_1+\alpha_2t_2)^n$ with non-symbolic coefficients $\alpha_0, \cdots, \alpha_n$. If $(\alpha t)^n$ is a perfect nth power $(t_1t)^n$, α will be the point on C^n of S_n whose parameter is t_1, or briefly the point t_1. Also if $(at)^n$ is a binary form, all points which satisfy the linear apolarity condition $(\alpha a)^n=0$ lie on the $S_{n-1}a$ with coordinates $a_0, \cdots, a_n$. The S_{n-p} $(t_1t)^p(\beta t)^{n-p}$, with parameters $\beta_0, \cdots, \beta_{n-p}$, is the osculating $(n-p)$-space O_{n-p,t_1} to C^n at t_1.† This notat on is helpful in the development of some of the properties of the normal rational n-ic curve. Many of the analogous properties for the case $n=5$ have been found by other methods by A. L. Hjelmann.‡

2. *The Axes of C^n.* An axis of C^n is a line which lies in $(n-1)$ O_{n-1}'s to C^n. The axes of C^n are given by

$$(\alpha t)^n = (t_1t)(t_2t)\cdots(t_{n-1}t)(st),$$

parameters s_0, s_1, the t_i being parameters of points of C^n.

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* Thus, although we have no actual example, it is conceivable that there exists, in E_5, a common boundary M of three domains D_k each of which is u.l.i-c. for $i=0, 1$. If so, $p^2(D_k)$ is infinite for $k=1, 2, 3$; and $p^3(E_5-M)$ is finite.

† Grace and Young, *The Algebra of Invariants*, 1903, Chapter 11.

‡ A. L. Hjelmann, *Sur les courbes gauches rationelles du cinquième ordre*, Annales Academiae Scientiarum Fennicae, (A), vol. 3 (1912–13), No. 11.