ON THE UNIVALENCY OF CESÀRO SUMS OF UNIVALENT FUNCTIONS*

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Introduction. Let

$$f(z) = z + \sum_{2}^{\infty} a_n z^n \tag{1}$$

be analytic and univalent for $|z| < 1$, that is, if z_1 and z_2 $(z_1 \neq z_2)$ are any two points inside the unit circle, then

$$f(z_1) - f(z_2) \neq 0.$$

The partial sums $S_n(z)$ of $f(z)$,

$$S_n(z) = z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n, \tag{2}$$

in general are not univalent in the unit circle, though they are univalent for $|z| < 1/4$ as G. Szegö has shown.‡ In some cases, however, one can say that the partial sums are univalent in the whole unit circle. J. W. Alexander§ has shown, for example, that if the coefficients of (1) are real and positive and such that the numbers na_n form a decreasing sequence, then not only $f(z)$ but all its partial sums are univalent in the unit circle.

To the best of the writer's knowledge the only results obtained to date regarding the univalency of the Cesàro sums of univalent functions are those of L. Fejér‖ who showed that if $f(z)$ is real on the real axis, and convex in the direction of the imaginary axis for $|z| < 1$, then all the Cesàro sums of the third order are univalent for $|z| < 1$.

In this paper we show that if the ordinary partial sums of

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‡ See G. Szegö, *Zur Theorie der schlichten Abbildungen*, Mathematische Annalen, vol. 100 (1928), pp. 188–211.

§ See J. W. Alexander, *Functions which map the interior of the unit circle upon simple regions*, Annals of Mathematics, (2), vol. 17 (1915–16), pp. 12–22.

‖ See L. Fejér, *Neue Eigenschaften der Mittelwerte bei den Fourierreihen*, Journal of the London Mathematical Society, vol. 8 (1933), pp. 53–62.

$f(x)$ are univalent for $|z|<1$ when $f(z)$ has this property, then also all the Cesàro sums of the first order are univalent in the unit circle.

For the proof of our theorem we make use of the following theorem due to N. Obrechkoff* which we submit as the lemma below.

LEMMA. *Let $P(z)$ be a polynomial of degree n whose zeros are all outside $|z|=1$, and let $Q(z)$ be a polynomial of degree m whose zeros are inside $|z|\leqq r<1$. If λ is an arbitrary positive number, then the polynomial*

$$2z[P'(z)\cdot Q(z)-\lambda P(z)\cdot Q'(z)]+(m\lambda-n)P(z)\cdot Q(z)$$

has no zeros in the annulus $r<|z|<1$.

In particular, if we take $Q(z)\equiv z$, then the polynomial $2z[zP'(z)-\lambda P(z)]+(\lambda-n)z\cdot P(z)$ has no zeros in $0<|z|<1$. In other words, $[P(z)-(2/(\lambda+n))zP'(z)]$ has no zeros in $|z|<1$.

Let

$$S_n(z)=z+a_2z^2+a_3z^3+\cdots+a_nz^n \tag{3}$$

be univalent in the unit circle. The necessary and sufficient condition for $S_n(z)$ to be univalent for $|z|<1$ is, as J. Dieudonné has shown,† that the zeros of the polynomial

$$P(z,\theta)\equiv 1+\sum_{k=2}^{n}a_k\frac{\sin k\theta}{\sin\theta}z^{k-1} \tag{4}$$

lie outside of the unit circle for all values of θ. If $S_n(z)$ is univalent for $|z|<1$, we know by our lemma that $[P(z,\theta)-(2/(\lambda+n-1))zP'(z,\theta)]$ has no zeros in $|z|<1$ or that

$$F(z,\theta)\equiv 1+\sum_{k=2}^{n}\left\{1-\frac{2(k-1)}{\lambda+n-1}\right\}a_k\frac{\sin k\theta}{\sin\theta}z^{k-1} \tag{5}$$

has no zeros inside $|z|<1$ for all θ. It follows by the theorem of Dieudonné, then, that the polynomial

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* See N. Obrechkoff, *Sur les racines des équations algébriques*, Tôhoku Mathematical Journal, vol. 38 (1933), p. 100.

† See J. Dieudonné, Annales de l'Ecole Normale, vol. 48 (1931), p. 310.

$$(6) \qquad f_n(z, \lambda) \equiv z + \sum_{k=2}^{n} \left\{ 1 - \frac{2(k-1)}{\lambda + n - 1} \right\} a_k z^k$$

is univalent in the unit circle for all $\lambda > 0$.

We denote the nth Cesàro sum of the first order of (3) by

$$(7) \qquad S_n^{(1)}(z) \equiv \frac{S_1(z) + S_2(z) + \cdots + S_n(z)}{n},$$

$$(8) \qquad \begin{aligned} S_n^{(1)}(z) &= z + \left(1 - \frac{1}{n}\right) a_2 z^2 + \cdots \\ &\quad + \left(1 - \frac{k-1}{n}\right) a_k z^k + \cdots + \frac{a_n}{n} z^n. \end{aligned}$$

If in (6) we give λ successively the values $n-1$, $n+1$, $n+3, \cdots$, we then see that the Cesàro sums of $S_n(z)$

$$S_{n-1}^{(1)}(z), \; S_n^{(1)}(z), \; S_{n+1}^{(1)}(z), \; \cdots$$

are univalent for $|z| < 1$ whenever $S_n(z)$ has this property. Thus we have the following theorem.

THEOREM. *If the polynomial*

$$S_n(z) = z + a_2 z^2 + \cdots + a_n z^n$$

is univalent in the unit circle, then the $(n+k)$*th Cesàro partial sum of the first order of* $S_n(z)$ *is also univalent in the unit circle for* $k \geqq -1$.

COROLLARY. *If all the partial sums* $S_n(z)$ *of* $f(z)$ *are univalent in the unit circle, then so also are all the Cesàro sums of the first order of* $f(z)$.

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