ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

82. Dr. Reinhold Baer: *The subgroup of the elements of finite order of an abelian group.*

Denote by $F(A)$ the subgroup of all the elements of finite order of the abelian group A. Then, if F is an abelian group whose elements are of finite order and J an abelian group without elements $\neq 0$ of finite order, the direct sum $A=F+J$ satisfies: $F(A)\backsim F$, $A/F(A)\backsim J$. $F+J$ is essentially the only group satisfying these conditions if, and only if, $F(A)$ is a direct summand of every abelian group A such that $F(A)\backsim F$, $A/F(A)\backsim J$. The pairs of groups F, J with this property are characterized by a simple relation between the structure of F and the structure of J. (Received January 24, 1936.)

83. Professor Clifford Bell: *On a theorem in higher plane curves.*

In Hilton's *Higher Plane Curves* the function $F(t)\equiv |f(t)\phi'(t)\psi''(t)|$ appears in a theorem concerning the cusps and inflections of a plane curve which is represented parametrically by the equation $x:y:z=f(t):\phi(t):\psi(t)$. The theorem states that if f, ϕ, ψ satisfy certain conditions, the cusps are given by the repeated roots of $F(t)=0$, while the inflections are given by the single roots. In this note, making use of the Plücker numbers, a proof of the theorem is given which leads to additional information about the repeated roots. (Received January 6, 1936).

84. Professor Salomon Bochner and Dr. W. T. Martin (National Research Fellow): *Singularities of composite functions in several variables.*

A generalization to functions in several variables is formulated for the classical theorem of Hadamard on the composition of singularities and stars of regularity. The generalization involves an alternative restatement of the classical theorem itself. (Received January 20, 1936.)

85. Professor A. D. Campbell: *A note on primitive roots in a Galois field.*

By using the condition that a primitive irreducible n-ic modulo p of the form $X^n-A_1X^{n-1}-A_2X^{n-2}-\cdots-A_{n-1}X-A_n=0$ must not divide X^m-1

$=0$ for $m<p^n-1$ the author derives a useful method for determining primitive roots of a Galois field of order p^n. (Received January 31, 1936.)

86. Professor Leonard Carlitz: *On higher congruences.*

This paper is concerned with congruences of the form $A_0t^{p^{nm}}+A_1t^{p^{n(m-1)}}+\cdots+A_mt\equiv B \pmod P$, where A, B, P are polynomials in an indeterminate x with coefficients in a Galois field of order p^n. For earlier results see abstract 41-9-336; also this Bulletin, vol. 41 (1935), pp. 907–914. In the present paper the homogeneous case ($B\equiv 0$) is assumed solved; criteria are then obtained for the solvability of the non-homogeneous case ($B\not\equiv 0$). The paper also contains some results on the homogeneous case as well as a new and simple proof of a criterion for the solvability of $t^{p^n}-t\equiv B \pmod P$. (Received January 24, 1936.)

87. Professor Leonard Carlitz: *On some arithmetic functions of several arguments.*

In this note are defined certain generalizations of the Möbius μ- and the Euler ϕ-functions. As an example, in the case of three variables, $\phi'''(m_1, m_2, m_3)$ is the number of triplets (u_1, u_2, u_3), $u_i \pmod{m_i}$, and $(u_1, u_2, u_3, m_1, m_2, m_3)=1$; $\phi''(m_1, m_2, m_3)$ is the number of triplets for which $(u_1, u_2, m_1, m_2)=1$ and two similar conditions hold. The functions were suggested by manipulating the series $\Sigma A(m_1, m_2, m_3)$ summed first over all m for which $(m_1, m_2, m_3)=1$, and secondly over all m for which $(m_2, m_3)=(m_3, m_2)=(m_1, m_2)=1$. (Received January 24, 1936.)

88. Professor Leonard Carlitz: *Some problems in additive arithmetic.*

This paper is concerned with sums of the type $\Sigma\alpha(m_1, \cdots, m_k)$ extended over all positive integral m_j for which $m=m_1+\cdots+m_k$; α is an arithmetic function of $k\geqq 2$ arguments and satisfies certain order conditions. In an earlier paper (Quarterly Journal of Mathematics, vol. 3 (1932), pp. 273–290) the case $\alpha(m_1, \cdots, m_k)\equiv\alpha(m_1)\cdots\alpha(m_k)$ was treated. Similar results are now obtained for general $\alpha(m_1, \cdots, m_k)$; the proof is no longer by induction but otherwise very much like the earlier proof. The results are applied to the sum $\Sigma\alpha(m_1, \cdots, m_k)$ extended over all positive m_j and primes p_i such that $n=m_1+\cdots+m_k+p_1+\cdots+p_s$. (Received January 24, 1936.)

89. Dr. J. A. Clarkson (National Research Fellow): *Uniformly convex spaces.*

Banach spaces are considered in which the norm is uniformly convex; expressed in geometrical language, the mid-point of a variable chord of the unit sphere in the space cannot approach the surface of the sphere unless the length of the chord goes to zero. All finite dimensional euclidean spaces, and Hilbert space, possess this property: it is shown that all spaces L_p and l_p are uniformly convex if p exceeds unity. A strengthened form of the ordinary triangular inequality is derived for such spaces. By means of this relation it is proved that functions of bounded variation from a euclidean space into a uniformly convex space are differentiable almost everywhere, that if such functions are absolutely continuous they are (Bochner) integrals of their derivatives, that a continuous rectifiable curve in a uniformly convex space has a tangent almost everywhere. (Received January 27, 1936.)

90. Mr. Nelson Dunford: *Integration of abstract functions.*

A sequence $f_n(P)$ of functions summable (in Bochner's or in the author's sense) on (0, 1) to the Banach space X may converge to $f(P)$ almost everywhere and one may have $\lim_n \int_e f_n(P)dP$ existing for every measurable set e without having $f(P)$ summable. An extended class of summable functions and an extended integral may be defined by putting $\int_e f(P)dP = \lim_n \int_e f_n(P)dP$. The integral is absolutely continuous and completely additive. The space of summable functions can be normed so as to be complete, and, in case X has a base, every function summable in the sense of Garrett Birkhoff is also summable in the extended sense to the same value. Every additive set function with values in a Banach space with a base which vanishes on sets of measure zero is absolutely continuous. (Received January 31, 1936.)

91. Mr. Nelson Dunford: *Linear operations.*

The general linear operator on L to $L_q(q>1)$ is given by the formula $T\phi = \int_0^1 K(P, Q)\phi(P)dP$ where the kernel satisfies the condition ess. $\sup_P \{\int_0^1 |K(P, Q)|^q dQ\}^{1/q} < \infty$. This number is the evaluation of $|T|$. T is completely continuous if and only if it satisfies the further condition $\lim_{h\to 0}$ ess. $\sup_P \int_0^1 |K(P, Q+h) - K(P, Q)|^q dQ = 0$. Analogous results hold with L replaced by the space of absolutely continuous functions, or with L_q replaced by a Banach space with a uniformly convex norm. If $f(P)$ on (0, 1) to $L_q(q \geqq 1)$ is measurable and if $\int_0^1 \| f(P) \|^{p/(p-1)} dP < \infty$, then the transformation $T\phi = \int_0^1 f(P)\phi(P)dP$ is completely continuous on L_p to L_q. This gives the author's (or Bochner's) integral a number of further properties in common with the Lebesgue integral of real functions. (Received January 31, 1936.)

92. Professor L. A. Dye: *Involutorial transformations associated with a ruled quartic surface with a double twisted cubic.*

A (1, 1) correspondence is established between the generators of a ruled quartic surface R which has a double twisted cubic and a simple line as directrices, and the surfaces of a pencil of quadrics. A general point P in space fixes a quadric and a generator which with P determines a plane tangent to R at a point Q. The line PQ meets the quadric in a residual point P' which is the image of P in involutorial transformation. Two cases are considered, one in which the simple directrix line is a part of the base of the pencil of quadrics, and in the other the twisted cubic lies on all surfaces of the pencil. The transformations are of order 29 and have some involved contact relations among the fundamental elements. (Received February 1, 1936.)

93. Professor H. T. Engstrom: *On substitution polynomials.*

A product of two polynomials $f(x)$ and $g(x)$ with coefficients in a field K is defined as $f(g(x))$. $F(x)$ is said to be irreducible if $F(x) \neq f(g(x))$ with $f(x)$

and $g(x)$ of degrees greater than 1. In decompositions of a polynomial into irreducible factors the number of factors is unique and their degrees are unique except for order. This theorem was proved by Ritt when K is the field of complex numbers. The author gives a purely algebraic proof which is valid for all fields K of characteristic 0. (Received February 1, 1936.)

94. Professor H. T. Engstrom and Dr. Max Zorn: *Geometry in a space-time of Page.* Preliminary report.

The authors show that the axioms of Page, which can be stated in essentially topological form, imply a definite structure for his "3-dimensional manifold of equivalent particle observers." The following results are obtained: 1. A description by coordinates; 2. The existence of an essentially unique time; 3. A metric admitting a transitive group of isometric mappings; 4. Ordinary spherical geometry. (Received February 1, 1936.)

95. Mr. Aaron Fialkow: *The geometry of single-parameter families of plane curves.*

Consider the single-parameter family of plane curves given by the equation $u(x, y)=$constant. Draw the orthogonal trajectories of these curves and study the geometry of the resulting orthogonal net under the assumption that $u(x, y)$ is a solution of one of the differential equations: (1) $u_{xx}+u_{yy}=f(u)$, (2) $u_{xx}u_{yy}-u_{xy}^2=f(u)$, (3) $u_{xx}u_y^2-2u_{xy}u_xu_y+u_{yy}u_x^2=f(u)$, (4) $u_{xx}(1+u_y^2)-2u_{xy}u_xu_y+u_{yy}(1+u_x^2)=0$. Through an arbitrary point of the plane there pass two perpendicular curves. At this point the curvature as well as the tangential and normal derivatives of the curvature are defined for each curve. The geometric properties of the net may be stated as equations in these intrinsic geometric quantities which are satisfied at each point. Complete geometric characterizations of this type for the four differential equations given above are obtained. There are also derived further conditions that $f(u)$ shall be a constant or zero in the first two equations. Some of these results may be interpreted as stating the necessary and sufficient conditions that a single-parameter family of curves be the projections of the ∞^1 parallel plane sections of a developable or minimal surface. (Received January 20, 1936.)

96. Dr. M. M. Flood: *The resultant matrix of two polynomials.*

If A is the square matrix of order n whose first row has the element $-a_k$ in the kth column, unity in the jth row and $(j-1)$st column for $j=2, 3, \cdots, n$, and zeros elsewhere, then the polynomial $f(x)=x^n+a_1x^{n-1}+\cdots+a_n$ is the characteristic function of A. If $g(x)$ is any other polynomial, the author calls $g(A)$ the *resultant matrix* of $g(x)$ and $f(x)$. Now if p is the degree of the HCF of $f(x)$ and $g(x)$, and if M_k, for $k=0, 1, \cdots, p$, denote the minors formed from $g(A)$ by using the last $n-p$ rows and first $n-p-1$ and $(n-p+k)$th columns, the principal theorem of this paper states that $M_0[g(x), f(x)]=M_0x^p+M_1x^{p-1}+\cdots+M_p$. This extends the theorem of W. V. Parker (American Mathematical Monthly, vol. 42 (1935), p. 164) which states only that the nullity of the resultant matrix of two polynomials is equal to the degree of their HCF. (Received January 18, 1936.)

97. Mr. R. H. Fox: *On the asymptotic distribution of analytic almost periodic functions.*

In this paper is discussed the asymptotic distribution function of an analytic almost periodic or related function $f(s)$ in a strip $\alpha \leqq \sigma \leqq \beta$ of the s-plane where $s=\sigma+it$. It is shown that $f(s)$ possesses an asymptotic distribution function in a strip $\alpha \leqq \sigma \leqq \beta$ if, for example, $f(s)$ is uniformly almost periodic there. In addition, a simple integral formula connecting the strip distribution function with the linear series of line distribution functions belonging to the σ of the interval $\alpha \leqq \sigma \leqq \beta$ is developed, as well as an analogous relation connecting their Fourier transforms. The results are then applied to the case of independent moduli. It is found, in particular, that if $\alpha > 1/2$ the logarithm of the Riemann-zeta function possesses a strip distribution function which is absolutely continuous with a density having continuous partial derivatives of all orders; all moments exist and belong to a determined moment problem. If, in addition, $\beta < 1$, the density is a transcendental entire function of its two variables. (Received January 29, 1936.)

98. Professor L. M. Graves: *Under-determined systems of linear differential equations.*

It has been pointed out by Carrus that the general solution of a system of ordinary linear differential equations may usually be obtained by explicit formulas not even involving quadratures, provided the number of equations is fewer than the number of unknown functions. When the system of equations is written as a system of first order equations, the method and results can be stated much more clearly and briefly, as well as more completely than was done by Carrus. It is shown also that the method is applicable to certain equations in which the operation of differentiation is replaced by much more general additive operators. Certain types of linear partial differential equations of the first and second orders are discussed as examples. (Received January 29, 1936.)

99. Mr. D. W. Hall: *On the non-alternating images of linear graphs.*

In this paper a study is made of the possible images of a linear graph under a non-alternating transformation (see G. T. Whyburn, American Journal of Mathematics, vol. 46 (1934), no. 2). It is shown that: I. A necessary and sufficient condition that a cyclic curve C be the non-alternating image of a linear graph is that C be the sum of a finite number of simple arcs; II. If B be be the non-alternating image of a linear graph A, then every true cyclic element of B is the sum of a finite number of simple arcs. (Received January 29, 1936.)

100. Dr. E. K. Haviland: *On an asymptotic expression for a certain integral.*

An exact proof is given for the asymptotic formula $J=\int_0^\infty \exp\,[\alpha x-\phi(x)]dx$

$\sim$exp $[\xi\phi'(\xi)-\phi(\xi)](2\pi/\phi''(\xi))^{1/2}$, where $\alpha=\phi'(\xi)$, under the conditions: (i) ϕ, ϕ', and ϕ'' are positive, continuous, and monotone increasing for non-negative x, and all three tend to infinity with x; (ii) ϕ''' exists and is positive if x is positive; (iii) ϕ'' and ϕ''' are such that ϕ'''/ϕ'' and ψ/ϕ'' tend to definite limits (including $\pm\infty$) as x becomes infinite, where ψ is a logarithmico-exponential function. It is shown by an example that if condition (iii) is not satisfied, the asymptotic formula need not hold. (Received January 16, 1936.)

101. Professor Edward Kasner: *A triangular inequality in conformal geometry.*

If three curves C_1, C_2, C_3 touch each other at one point, there are three horn angles C_1C_2, C_2C_3, C_3C_1. The author discusses the conformal invariants and obtains an inequality relation. The absolute invariant of a horn angle as previously defined is $I(1, 2)=(\delta_2-\delta_1)/(\gamma_2-\gamma_1)^2$ where γ means curvature and $\delta=d\gamma/ds$. The fundamental theorem is $I(12)I(13)+I(23)I(21)+I(31)I(32)\geqq 0$. The *measure* of the horn angle is defined as $M(1, 2)=$reciprocal of $I(1, 2)=(\gamma_2-\gamma_1)^2/(\delta_2-\delta_1)$. The result is then $M(1, 2)+M(2, 3)+M(3, 1)\geqq 0$ provided $M(1, 2)M(2, 3)M(3, 1)<0$, that is, provided $(\delta_2-\delta_1)(\delta_3-\delta_2)(\delta_1-\delta_3)<0$. The equality sign holds only when the three curves are related so that the differences of the three δ's are proportional to the differences of the three curvatures γ. In particular this is valid when C_3 is obtained from C_1, C_2 by successive Schwarzian reflexion, or by conformal bisection as defined in the Proceedings of the International Congress of Mathematicians, Cambridge, 1912, p. 81. (Received January 30, 1936.)

102. Mr. R. B. Kershner: *On the addition of convex curves.*

Bohr (1913) has shown that the vectorial sum of n convex curves C_j is either a closed convex region or the ring-shaped domain between two convex curves. It is known that the supporting function of the outer curve is the sum of the supporting functions $h_j(\theta)$ of the C_j. In the present paper it is shown that if the inner curve exists then there must be one curve, say C_1, such that the supporting function of the inner curve is $h_1(\theta)-\sum_{i=2}^{n}h_j(\theta+\pi)$ save at most for the θ-intervals corresponding to the corners (if any) of the inner curve. This explicit result makes possible an analytical discussion of the inner curve similar to that given by Haviland (1933) for the outer curve. (Received January 22, 1936.)

103. Dr. D. H. Lehmer: *On the converse of Fermat's theorem.*

It has been known for more than a century that the statement (a) "n is a prime if it divides 2^n-2" suffers exceptions. In fact an infinite number of exceptions exist. Additional conditions on n so that (a) will be true have been given, but the best of these require the knowledge of a prime factor (the larger the better) of $n-1$. Another method of making (a) available as a test for primality, at least for a certain range of numbers, is to list all exceptions to (a) lying in this range together with their factorization. However, many of these exceptions are multiples of small primes so that their composite nature is readily discovered, and these may be disregarded. The writer has prepared a complete list of these exceptions to (a) between 10^7 and 10^8 whose least prime factor exceeds 313, together with this factor. This list contains some 500 numbers and will appear in the American Mathematical Monthly. Twenty minutes suffice for determining by this method the prime or composite character of any 8-digit number. (Received January 25, 1936.)

104. Dr. D. H. Lehmer: *Polynomials for the n-ary composition of numerical functions.*

This paper deals with polynomials $\psi(x_1, x_2, \cdots, x_n)$ in n positive integers whose values are positive integers and which represent all positive integers, and are such that $\psi(x_1, x_2, \cdots, x_{n-1}, \psi(x_n, \cdots, x_{2n-1}))$ is a symmetric function of all of its $2n-1$ variables. Such polynomials would form a basis of the simplest sort of composition of numerical functions n at a time. All such polynomials are found and they constitute only a one parameter family. Moreover, each polynomial is the result of $n-2$ iterations of an appropriately chosen polynomial in 2 variables. This shows that n-ary composition with respect to polynomials may be achieved by repeated applications of the ordinary binary composition. (Received January 28, 1936.)

105. Mr. W. T. Puckett and Mr. G. C. Watson: *Concerning images of certain continua under monotone transformations.*

Let the continuum A having no continuum of condensation be decomposed into $A = F + \Sigma A_i$, where F is a closed and totally disconnected set and each A_i is a free arc having its endpoints in F. If $T(A) = B$ is a monotone transformation, then B takes the form $B = T(F) + \Sigma B_i$, where each B_i is a free arc having its endpoints in $T(F)$. Whence any continuum of condensation of B must be contained in $T(F)$. In case A is a linear graph the number of arcs B_i of B does not exceed the number of arcs A_i of A. In general, if, for any $b \epsilon B$, $T^{-1}(b)$ contains no simple closed curve and at most one point of F, then B is homeomorphic with A. (Received January 29, 1936.)

106. Mr. E. D. Rainville: *On heat conduction in wedge shaped dams.*

Ordinarily temperature studies of concrete dams are based on approximations which replace the actual shape of the dam by a semi-infinite solid or by a slab of finite width. Often a wedge shaped figure is nearer to the actual case. In this paper certain problems of heat conduction in an infinite plane wedge are solved in two ways, one of which may be based upon Carslaw's point source function for such a wedge. A mean of practical importance is obtained and the formulas are simplified for computation purposes. The boundary and initial conditions satisfied are those best suited to the study of temperatures in many concrete dams. Only a finite portion of the wedge represents the dam; the remainder is treated as bedrock. An application to Boulder Dam is included. (Received January 10, 1936.)

107. Professor Francis Regan: *A note on a preceding paper concerning the admissibility of time series.*

In a paper (Transactions of this Society, vol. 36 (1934), no. 3) a lemma and a theorem due to Copeland are used in the development of the problem. In this note these two theorems are extended to apply in the field of geometrical probability. (Received January 15, 1936.)

108. Mr. L. B. Robinson: *A contribution to the theory of pseudo-functions.*

Recently the author proved that the solution of the Izumi equation $u'(x)=a(x)u(x^k)$, $(k=2, 3, 4, \cdots)$, was a pseudo-function, provided that $a(x)$ is an integral function with positive coefficients. He has succeeded now in demonstrating his theorem after lifting the restriction on $a(x)$. (Received January 24, 1936.)

109. Dr. Abraham Sinkov: *Necessary and sufficient conditions for generating the simple groups of order* 504 *and* 1092.

Necessary and sufficient conditions for generating the simple groups of orders 60, 168, and 660 by two operators of periods two and three are already known. No such definition is possible for the simple group of order 360. The purpose of the present study is to determine necessary and sufficient conditions for the two remaining simple groups whose orders do not exceed 1092. (Received January 21, 1936.)

110. Professor Morgan Ward and Mr. F. B. Fuller: *Continuous iteration of real functions.*

By an extension of a method developed by one of the authors (Ward, this Bulletin, vol. 40 (1934), no. 10) they obtain all the real continuous iterations of a given real continuous steadily increasing function $E(x)$ of the real variable x defined in the range $-\infty < a \leqq x < \infty$. By a continuous iteration of $E(x)$ is meant a real function $E_y(x)$ of the two real variables x and y such that $E_0(x)=x$, $E_1(x)=E(x)$; $E_{y+z}(x)=E_y(E_z(x))$; $E_y(a)$ steadily increasing and continuous in y in the range $0 \leqq y \leqq 1$. The iteration is completely determined by the choice of $E_y(a)$ for $0 \leqq y \leqq 1$, and the authors give the simple formulas which together enable them to exhibit $E_y(x)$ explicitly for $x \geqq a$, $y \geqq 0$. (Received January 11, 1936.)

111. Mr. R. Wilson: *A note on the asymptotic properties of orthogonal polynomials.*

Consider the system of orthogonal Tchebycheff polynomials $\phi_n(x)=x^n + \cdots$ corresponding to the finite interval (0, 1), and to a characteristic "S-function" $p(x)$. Making use of the recurrence-relation for $\phi_n(x)$, and of Poincaré's theorem, as supplemented by Perron, the author proves that $\limsup_{n\to\infty} [\phi_n(x)]^{1/n}$ for $0 \leqq x \leqq 1$ is 1/4, which is transfinite diameter of (0, 1). This supplements results announced earlier by J. Shohat for another type of $p(x)$ (American Journal of Mathematics, vol. 55 (1933), pp. 218–230). It follows that any function $f(x)$, analytic on (0, 1), can be expanded in a series according to the polynomials $\{\phi_n(x)\}$, which converges absolutely and uniformly for $0 \leqq x \leqq 1$. (Received January 9, 1936.)

112. Professors Einar Hille and Otto Szász: *On the completeness of Lambert functions.*

Let $k_\beta(w)=\sum_1^\infty n^\beta w^n$, and let $S(\alpha,\ \beta,\ \lambda)$ denote the set of functions $(1-t)^\alpha k_\beta(t^{\lambda_n})$ where the λ's are complex numbers such that $\Re(\lambda_n)\geqq\delta>0$, and $\sum_1^\infty\Re(1/\lambda_n)=+\infty$. This set is complete in $L_p(0,\ 1)$, $1\leqq p<\infty$, if $\alpha>\max(\beta+1-1/p,\ -1/p)$. The set $S(\alpha,\ \beta,\ \lambda)+1$ is complete in $C[0,\ 1]$ if $\alpha>\max(\beta+1,\ 0)$. The analysis is based upon a use of the inversion formula of Möbius, and is capable of extensive generalizations. (Received February 17, 1936.)

113. Dr. Walter Leighton: *A continued fraction expansion.*

To every function $f(z)$ analytic in a region $R\supset z=0$ there corresponds a unique continued fraction expansion of the form (1): $c_0+[c_n z^{a_n}/1]_{n=1}^{n=\infty}$, where the a_n are positive integers and the c_n are constants. If $f(z)$ is a rational function of z, the continued fraction is terminating. If $f(z)$ is not rational, $c_n\neq 0$ $(n=1,\ 2,\cdots)$. The development (1) is the natural generalization of the classical expansion (2): $c_0+[c_n z/1]_{n=1}^{n=\infty}$ and avoids the introduction of the cumbersome conditions, inherent in (2), that $f(z)$ be "normal." Many of the classical theorems concerning (2) have valid analogues in the theory of (1). Several new convergence criteria are obtained. Example: If lim sup $|c_{2n+1}|=b_1$, lim inf $|c_{2n}|=b_0$, and $25b_1<b_0$, the continued fraction (2) converges except for poles in the annular region $b_1^{-1}>|4z|>25b_0^{-1}$. (Received February 25, 1936.)

114. Mr. R. P. Boas: *Some theorems on Fourier transforms and conjugate trigonometric integrals.*

Writing $f^*(x)=(1/\pi)\int_0^\infty[f(x+t)-f(x-t)][(\sin\ t)/t^2]dt$, the author shows that a necessary and sufficient condition that $f(x)\epsilon L^2\ (-\infty,\ \infty)$ should have a Fourier transform vanishing almost everywhere on $(-1,\ 1)$ is that $f^{**}(x)=-f(x)$ almost everywhere. He also considers $f^*(x)$ for a trigonometric Stieltjes integral, $f(x)=\int_{-R}^R e^{ixt}d\alpha(t)$, $\alpha(t)$ of bounded variation. If $\alpha(t)$ is constant on $(0,\ 1)$ and on $(-1,0)$, $f^*(x)=-\bar f(x)$, where $\bar f(x)$ is the conjugate trigonometric integral. This relation yields an inequality for the conjugate of a trigonometric integral of the special form considered: $|\bar f(x)|<(2/\pi)(2+\log\ R)$ if $|f(x)|\leqq 1$; $2/\pi$ is the best possible constant. This generalizes a result of M. Fekete for trigonometric polynomials. There is also obtained for trigonometric integrals (with $\alpha(t)$ not specialized on $(-1,\ 1)$) a generalization of the extension by G. Szegö (Schriften der Königsberger gelehrten Gesellschaft, vol. 5 (1928), pp. 59–70) of S. Bernstein's inequality for the derivative of a trigonometric polynomial: $R|\sigma(x)|+[f'(x)^2+\bar f'(x)^2]^{(1/2)}\leqq R$ if $|f(x)|\leqq 1$, where $\sigma(x)$ is a certain arithmetic mean of $f(x)$. Similar inequalities involving mean values, suggested by a result of A. Zygmund, are also obtained. (Received February 20, 1936.)

115. Dr. Nathan Kaplan: *Einstein hypersurfaces of a euclidean space.*

In this paper, the properties of an Einstein hypersurface of m dimensions (V_m) which is immersed in an ordinary euclidean space of $m+1$ dimensions (R_{m+1}) are studied. The question of the existence of these hypersurfaces is left open. The following results are obtained: (1) this hypersurface can have at most two distinct curvatures (l) and (k); (2) these are related by $(x-1)l+(m-x-1)k=0$ where (x) and ($m-x$) are the number of curvatures equal to (l) and (k) respectively; (3) the (x) principal directions having (l) as curvature build a subspace V_x; (4) a similar result is valid for the remaining ($m-x$) curvatures; (5) l and k are constants; (6) the subspaces V_x and V_{m-x} are of constant curvature and geodesic in V_m; and (7) these principal directions can always be selected such that they form an m-tuply orthogonal net. (Received February 20, 1936.)

116. Professor Marston Morse and Dr. Walter Leighton: *Singular quadratic functionals.*

The authors study the integral of a quadratic form in y' and y with coefficients which are continuous functions of x on an open interval, with singularities of various types at the ends of the interval. The coefficient of y'^2 is assumed positive. The study is made in the medium of Lebesgue integral theory and leads to important special cases of singular differential equations of the regular type. A new definition of conjugate points is required and new necessary and sufficient conditions. The results have applications in quantum mechanics and include as special cases certain fundamental integral inequalities found in chapter VII of the recent book of Hardy, Littlewood, and Pólya. (Received February 26, 1936.)

117. Professor V. G. Grove: *Differential geometry of a surface at a planar point.*

A point of a surface is called a planar point of the third order if every curve on the surface through the point has an inflexion at the point. The tangents to the curve of intersection of the surface and its tangent plane at the point determine a cubic involution. The author finds a canonical expansion of the surface in which one edge of the tetrahedron of reference giving rise to the expansion is a non-specialized tangent, and a canonical expansion in which two edges of the tetrahedron are the Hessian lines of the cubic involution. By means of the osculants of Bompiani for a plane curve at a point of inflexion, he discusses the various neighborhoods of arbitrary plane sections of the surface through the non-specialized tangent, and of sections through the Hessian lines of the cubic involution. Finally the loci of certain points connected with Bompiani's osculants for all sections of the surface at the planar point are discussed. (Received February 17, 1936.)

118. Dr. P. W. Ketchum: *On the expansion of a function holomorphic in distinct regions.*

A function $f(z)$ analytic at the distinct points $a_1, a_2, \cdots, a_n$, but not necessarily analytic in any region including two of these points, has a unique expansion in the series $\sum_{m=1}^{\infty} c_m F_m(z)$ where the functions $F_m(z)$ are polynomials defined by the equations $F_{\nu n+k}(z)=\Pi_{i=1}^{n}(z-a_i)^{\nu+1}/(z-a_k)$, $(k=1, 2, \cdots, n)$. This series converges absolutely and uniformly to $f(z)$ in and on any lemniscate $\Pi_{i=1}^{n}|z-a_i|=\eta$ which consists of n distinct ovals in each of which $f(z)$ is analytic. Generalizations of this theorem are also given. Proofs are very simple, involving only a transformation of Bürmann's series. (Received February 17, 1936).

119. Professor Tibor Radó: *On transformations in the plane.* I *and* II.

If $u(x, y)$, $v(x, y)$ are single-valued and continuous in a closed square S, then the equations $u=u(x, y)$, $v=v(x, y)$ define a continuous transformation of S. Schauder (*Über stetige Abbildungen*, Fundamenta Mathematicae, vol. 12) obtained important results concerning the continuous transformations which are absolutely continuous in the sense of Banach. For the special case of transformations which satisfy a Lipschitz condition, Schauder derived a formula for the transformation of double integrals which has been applied recently in many investigations. On the other hand, for general absolutely continuous transformations Schauder had to make the additional restriction that a certain function $i(P)$ is bounded. The purpose of this paper is to remove this restriction and hence to make absolutely continuous transformations more easily manageable. The paper consists of two parts. The first part (*On continuous transformations*) contains a discussion of general continuous transformations and a proof that the restriction mentioned above is superfluous. In the second part (*On absolutely continuous transformations*) the formula for the transformation of double integrals is established, for absolutely continuous transformations, in complete generality, and semi-continuity properties are derived for certain integrals containing generalized Jacobians. The author plans to apply these results to the study of the area of surfaces in subsequent papers. (Received February 17, 1936.)

120. Professor R. D. Carmichael: *Proof that every positive integer is a sum of four squares.*

By means of certain reciprocal relations (essentially algebraic identities), which are of interest in themselves, the author gives a new proof of the stated theorem. The earlier proofs to which this one is most closely related are those of Euler and Dickson. (Received February 19, 1936.)

121. Professor R. D. Carmichael: *On numbers of the form* $a^2+\alpha b^2$.

Two elementary theorems concerning numbers of the given binary quadratic form are proved by very simple means and several elementary classic results are shown to follow immediately from these theorems. (Received February 19, 1936.)

122. Professor R. D. Carmichael: *On non-homogeneous linear differential equations of infinite order with constant coefficients.*

Using methods suggested by the stimulating memoir of A. Hurwitz (Acta Mathematica, vol. 20 (1897), pp. 285–312) and applying them to the indicated differential equations, the author obtains a series of results implying the existence under suitable hypotheses of several classes of integral solutions. (Received February 19, 1936.)

123. Professor R. D. Carmichael, Dr. W. T. Martin (National Research Fellow), and Dr. M. T. Bird: *On a classification of integral functions by means of certain invariant point properties.*

In the investigation of linear differential equations of infinite order and of certain other types of linear functional equations, the authors have severally had occasion to employ certain point properties of integral functions which remain unaltered in passing from one regular point to another. These invariant point properties are of considerable interest in themselves, and the theorems to which they give rise extend classic results in the theory of integral functions. In this paper the authors lay the foundations of a general theory of these invariant point properties and present the principal results which they have severally drawn upon in their recent investigations. (Received February 19, 1936.)

124. Professor Morris Marden: *An analogue to the Fekete theorem for the critical points of Green's function.*

Following a course parallel to that of his previous note on an analogue to the Grace-Heawood theorem (see abstract 42-1-72), the author finds the following as his principal result. "Let R_1 and R_2 be any two infinite regions with finite, non-intersecting boundaries B_1 and B_2, such that $G_1(x, y)$ and $G_2(x, y)$, the Green's functions (pole at ∞) for R_1 and R_2 respectively, have the same critical points. Let C_1 (center α_1, radius r_1) be any circle enclosing a closed branch of B_1 and let C_2 (center α_2, radius r_2) be any circle not overlapping C_1 but containing a closed branch of B_2. Then, if no critical point of the function $G(x, y) = [G_1(x, y) + G_2(x, y)]/2$ lies in the closed exterior of the hyperbola with foci at α_1 and α_2, and transverse axis of length $(r_1 + r_2)$, at least one critical point of $G(x, y)$—not a critical point of $G_1(x, y)$ and $G_2(x, y)$—lies in each interior of this hyperbola." This note will be published in combination with the previous one. (Received February 24, 1936.)

125. Professor Tibor Radó: *A lemma on the topological index.*

Let $C_k (k=0, 1, 2, \cdots)$ be a sequence of closed continuous curves given by equations $x = x_k(t)$, $y = y_k(t)$, where $x_k(t)$, $y_k(t)$ are continuous in $0 \leqq t \leqq 1$ and satisfy $x_k(0) = x_k(1)$, $y_k(0) = y_k(1)$. Let $n_k(x, y)$ be defined as follows: If the point (x, y) is not on C_k, then $n_k(x, y)$ is equal to the topological index of (x, y) with respect to C_k; otherwise $n_k(x, y) = 0$. Suppose all the functions $y_k(t)$ are of bounded variation in $0 \leqq t \leqq 1$ and denote by T_k the total variation of $y_k(t)$. Then we have the following lemma: If $x_k(t) \to x_0(t)$, $y_k(t) \to y_0(t)$ uniformly in $0 \leqq t \leqq 1$, and if $T_k \to T_0$, then $\iint |n(x, y) - n_k(x, y)| \, dx \, dy \to 0$. The paper concludes with a discussion of some corollaries and special cases, interesting either on account of their geometrical form or on account of applications to situations involving Jacobians. These applications will be developed in subsequent papers. (Received February 28, 1936.)

126. Professor Tibor Radó: *A remark on the area of surfaces.*

Let the continuous surface S be given by equations $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, $0 \leqq u \leqq 1$, $0 \leqq v \leqq 1$. Denote by $L(S)$ the area of S in the sense of

Lebesgue. For various special classes of surfaces it has been shown that $L(S)$ is given by the familiar integral formula. The most general result in this direction was obtained by McShane and Morrey. The purpose of this paper is to call attention to the following generalization of their result. In the definition of surfaces of class L (see Morrey, *A class of representations of manifolds*, American Journal of Mathematics, vol. 55 (1933)) replace condition (i) of Morrey ($x(u, v)$, $y(u, v)$, $z(u, v)$ absolutely continuous in the sense of Tonelli) by the assumption that $x(u, v)$, $y(u, v)$, $z(u, v)$ are of bounded variation in the sense of Tonelli. Call class L^* the class of surfaces determined by the definition thus modified. Then the area $L(S)$ of every surface S of class L^* is given by the familiar double integral (provided, of course, that we use for S a parametric representation with the properties specified in the definition of surfaces of class L^*). The proof depends upon certain simple properties of the topological index. (Received February 28, 1936.)

127. Dr. M. S. Webster: *On the relation of Appell polynomials to orthogonal polynomials.*

This note consists of a new, short proof of the recently established theorem that the system of Hermite polynomials is, except for a linear transformation on x, the only system of Appell polynomials which is also orthogonal. The proof is based upon the fundamental recurrence relation for orthogonal polynomials and the recurrence relation for Appell polynomials obtained by Sheffer. (Received February 28, 1936.)

128. Professor H. P. Thielman: *A generalization of the elementary transcendental functions by means of a theorem of Volterra.*

In this paper use is made of a theorem of Volterra for the derivation of transcendental functions from rational functions which vanish at infinity. The elementary transcendental functions are particular cases of the transcendentals so obtained. By applying the same process of transformation to these generalized transcendentals as was used for their derivation from rational functions, a larger class of transcendental functions is obtained, particular cases of which are Bessel's and Kelvin's functions of any order, and hypergeometric series. Many properties of these generalized transcendentals are then derived from the properties of the simpler functions which gave rise to them. The Laplace transformation is shown to be the inverse of one of the Volterra transformations here used. On the basis of this fact the asymptotic behavior of Bessel's and Kelvin's functions is studied and various infinite integrals of these functions are evaluated. (Received March 5, 1936.)

129. Dr. Rufus Oldenburger: *Ranks of 4-way matrices.*

With a given 4-way matrix $A=(a_{ijkl})$, $i, j, k, l=1, \cdots, n$, are associated 32 ranks which are defined in terms of the vanishing and non-vanishing of various determinant minors of A and matrices associated with A. These ranks divide into 6 essentially distinct sets according to the types of determinants in terms of which they are defined. In this paper there is determined the minimum value of k for each set such that when k of the ranks in a given set are unity the remainder are also unity. It is also shown that when certain ranks of a set are unity ranks in other sets must also be unity. The paper was suggested by a property studied by Hitchcock and the author which states that if a 3-way rank of a 3-way matrix is unity, one of the 2-way ranks of this matrix is also unity. The ranks of the present paper are invariant under non-singular linear transformations in any field, and are useful in studying canonical quatrilinear forms. (Received March 6, 1936.)

130. Dr. Rufus Oldenburger: *Arithmetic invariants of binary cubic and binary trilinear forms.*

The canonical binary cubic forms x^3, x^3+y^3, x^2y of the binary cubic are well known. These cubics have been classified by means of algebraic invariants. In the present paper it is shown that these cubics can be distinguished more simply by means of a single arithmetic invariant called "factorization rank." For the above forms this rank has the values 1, 2, and 3 respectively. The rank is defined in terms of the orders of 2-way matrices which are factors of a 3-way matrix associated with the given cubic. This rank is invariant under non-singular linear transformations on the given cubic in any field. In this paper there are also evaluated the factorization ranks of the canonical binary trilinear forms in the complex field. (Received March 6, 1936.)

131. Dr. S. F. Barber: *Genus of an algebraic variety.*

Let V_r be an irreducible algebraic variety in a linear space S_{r+1} with a double variety D_{r-1} as its only singularities. Further, let V_r be a generic projection of a variety W_r in S_d, free from singularities. F. Severi in Rendiconti del Circolo Matematico di Palermo, vol. 28 (1909) has given a definition of arithmetic genus of V_r and a second definition of arithmetic genus of W_r and proved them to give the same value for $r=3$. This paper proves them equal for any r. (Received March 4, 1936.)

132. Mr. G. D. Nichols: *A generalized element of decomposition for doubly periodic functions.*

Starting from the integral first used by Teixeira, a derivation is given for a generalized element of decomposition for doubly periodic functions of the first, second, and third kinds. Relations between special cases of this element and the classical ones of Hermite and Appell are given. (Received March 5, 1936.)