ON CERTAIN VARIETIES WHOSE CURVE SECTIONS ARE HYPERELLIPTIC CURVES*

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The properties of hyperelliptic curves, which have been described by Bobek,† are well known. One important property of such a curve, which must be of genus greater than unity, is that it carries on it one and only one complete and special series of groups of two points. From this property various others follow. For example, a hyperelliptic curve in any space can be transformed into a plane curve of order n with an $(n-2)$-fold point. Now surfaces in a space of any number of dimensions whose sections by spaces of dimension one lower are hyperelliptic curves have certain properties and these have been investigated by Castelnuovo.‡ Such a surface must contain ∞^1 conics such that through each point of it passes one and only one of them. It can be transformed or projected into one, of order n, in a 3-dimensional space having an $(n-2)$-fold line; and it is rational.

In this note we call attention to two varieties, in a higher space, whose curve sections are, as we shall show, hyperelliptic curves. One is the V_n^{2n+1} in S_{2n+1} which was the subject of an investigation by Babbage§ and the other is the $V_k^{2n-2k+1}$ in S_n which is the residual intersection of $(n-k)$ cubic hypersurfaces having in common the intersection M_{n-2}^4 of two quadric hypersurfaces of S_n. We shall describe in some detail the surface $V_k^{2n-2k+1}$ for $k=2$ or F^{2n-3} and also its projections in a 3-space. Incidentally, we shall obtain a property concerning linear spaces

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† Bobek, *Ueber hyperelliptische Curven*, Mathematischen Annalen, vol. 29 (1887), pp. 386–412.

‡ Castelnuovo, *Sulle superficie algebriche le cui sezioni piane sono curve iperellittiche*, Rendiconti del Circolo Matematico di Palermo, vol. 4 (1890), pp. 73–88.

§ Babbage, *A series of rational loci with one apparent double point*, Proceedings of the Cambridge Philosophical Society, vol. 27 (1931), pp. 399–403. See also B. C. Wong, *On a certain rational* V_n^{2n+1} *in* S_{2n+1}, American Journal of Mathematics, vol. 56 (1934), pp. 219–224.

contained in a general variety whose curve sections are hyperelliptic curves.

We may readily infer that any variety, V_t, of t dimensions in an r-space S_r with hyperelliptic curve sections must contain a rational ∞^1-system of quadric $(t-1)$-dimensional varieties such that through each point of it passes one and only one of them. A general S_{r-t+1} of S_r meets each of these quadric varieties in a pair of points and the ∞^1 pairs of points so obtained form a series of groups of two points on the curve in which S_{r-t+1} meets V_t. We see also that the variety is rational, in the sense that the coordinates of a point on it are rational functions of t non-homogeneous parameters, $t-1$ of which are the parameters of a point on one of the quadric varieties contained in V_t and the other is that of the variable quadric variety of the ∞^1-system.* Any section V_h for $h \leqq 2$ of the variety is also rational in this sense. Again, the variety can be transformed or projected into one, of order n, in a $(t+1)$-space S_{t+1} with an $(n-2)$-fold $(t-1)$-space so that any section of the projected variety by a 3-space of S_{t+1} will be a surface with an $(n-2)$-fold line.

Now we derive a property concerning linear spaces contained in V_t. It is known† that a quadric variety of $t-1$ dimensions in a t-space contains $\infty^{N_{m,t-1}}$ m-spaces, where

$$N_{m,t-1} = \frac{1}{2}(m+1)(2t-3m-2),$$

and

$$m \leqq \frac{1}{2}(t-2), \text{ if } t \text{ is even}; \qquad m \leqq \frac{1}{2}(t-1), \text{ if } t \text{ is odd}.$$

Since V_t contains ∞^1 such quadric varieties, it contains $\infty^{N_{m,t-1}+1}$ m-spaces. If t is even and $m=(t-2)/2$ and therefore $t=2m+2$, we have $\infty^{(m^2+3m+4)/2}$ m-spaces on V_{2m+2}; and, if t is odd and $m=(t-1)/2$ and therefore $t=2m+1$, we have

* We do not know whether every such V_t can be mapped upon a t-space.

† Bertini, *Projektive Geometrie Mehrdimensionaler Räume*, 1924, pp. 140–141.

$\infty^{(m^2+m+2)/2}$ m-spaces on V_{2m+1}. Thus, V_2 contains ∞^2 points; V_3 contains ∞^2 lines;* V_4 contains ∞^4 lines; and so on.

Let us now consider the varieties, V_n^{2n+1} in S_{2n+1} and $V_k^{2n-2k+1}$ in S_n, already mentioned above. The first one, V_n^{2n+1} in S_{2n+1}, as was shown by Babbage, can be represented upon an S_n by means of the cubic hypersurfaces of S_n passing through the intersection M_{n-2}^4 of two quadric hypersurfaces of S_n. It can be shown without difficulty that any $n-k$ of the cubic hypersurfaces intersect in a $V_k^{2n-2k+1}$ which has a $V_{k-1}^{4(n-k)}$ in common with M_{n-2}^4. To $V_k^{2n-2k+1}$ corresponds a section, V_k^{2n+1}, of V_n^{2n+1} by an S_{n+k+1} and to $V_{k-1}^{4(n-k)}$ corresponds a ruled variety $V_k^{4k(n-k)}$ on V_n^{2n+1}. Now V_n^{2n+1}, as was shown by Babbage, has ∞^1 quadric $(n-1)$-dimensional varieties each of which corresponds to a quadric hypersurface in S_n passing through M_{n-2}^4. Hence, V_n^{2n+1}, and therefore any section V_k^{2n+1} of it, has hyperelliptic curve sections. We infer that the $V_k^{2n-2k+1}$ in S_n to which V_k^{2n+1} corresponds must also have such sections. This result also follows from the fact that any quadric hypersurface in S_n passing through M_{n-2}^4 meets $V_k^{2n-2k+1}$ in a quadric $(k-1)$-dimensional variety besides the $V_k^{4(n-k)}$ which is on M_{n-2}^4. Then, $V_k^{2n-2k+1}$ has ∞^1 quadric $(k-1)$-dimensional varieties such that each point on it is on one of them.

The characteristics of V_n^{2n+1} are known,† and those of a surface section, F^{2n+1}, by an S_{n+3} can be easily calculated. The projection of F^{2n+1} in an S_4 has $(n-1)^2+(n-2)^2$ improper double points and the projection in S_3 has a double curve of order $n^2+(n-1)^2$ upon which lie $8(n-1)$ pinch points and $(2n-3)(2n^2-6n+7)/3$ triple points.

The surface, F^{2n-3}, of intersection of $n-2$ cubic hypersurfaces of S_n passing through M_{n-2}^4, to which corresponds the surface of the preceding paragraph, has hyperelliptic sections. According to Castelnuovo, it can be represented upon a plane f by an ∞^n-system of n-ic curves having one $(n-2)$-fold base point, A, and $2n-1$ simple points, B_i, $(i=1, 2, \cdots, 2n-1)$. From this representation we see that the surface is of class $8n-12$, that

* We do not know whether any 3-dimensional variety that has rational surface sections contains ∞^2 lines. A surface with rational curve sections is ruled and, in general, a V_t with rational curve sections is the locus of ∞^1 $(t-1)$-spaces; but a V_t with rational surface section is not the locus of ∞^2 $(t-2)$-spaces.

† B. C. Wong, *On a certain rational* V_n^{2n+1} *in* S_{2n+1}, loc. cit.

its projection in an S_4 has $2(n-3)(n-4)$ improper double points, and that its projection in an S_3 has a double curve of order $2(n-2)(n-3)$ with $8n-24$ pinch points. The image in f of the double curve is a curve of order $(n-3)(2n-1)$ passing through A $(n-3)(2n-5)$ times and through the $2n-1$ points B_i each $2(n-3)$ times.

The surface F^{2n-3} in S_n may be regarded as the projection of an F^{4n-4} in an S_{3n-1} from an S_{2n-2} determined by $2n-1$ general points upon it. The projection of this F^{4n-4} in S_4 has $8n^2-31n+31$ improper double points and the projection in S_3 has $16n-28$ pinch points and a double curve of order $8n^2-23n+17$. F^{4n-4} is normal in S_{3n-1} as it is representable upon f by the ∞^{3n-1}-systems of n-ic curves having one $(n-2)$-fold base point at A and no other base points.

If we project F^{2n-3} in S_n upon an S_3 of S_n from an S_{n-4} determined by $n-3$ general points on it, we have for projection an F^n with an $(n-2)$-fold line which constitutes the double curve of order $(n-2)(n-3)/2$. On this $(n-2)$-fold line are $4n-12$ pinch points.

In order to see better how this $(n-2)$-fold line on F^2 arises from projection, we consider the F^{n+2} in S_5 which is the projection of F^{2n-3} from $3n-6$ points on it. Its representation in f is effected by means of the ∞^5 n-ic curves passing through a given point A $n-2$ times and through each of $3n-6$ other given points $B_1, \cdots, B_{3n-6}$ once. There are ∞^2 curves of order $n-1$ in f having A for $(n-3)$-fold point and $B_1, \cdots, B_{3n-6}$ for simple points. Each of these curves goes into a curve, K^n, of order n on F^{n+2}, which, constituting with every conic of the surface a 4-space section of the surface, is a 3-space curve. Hence, there are ∞^2 such 3-space curves on F^{n+2}. Each of these curves lies on a quadric surface and meets the generators of one regulus $n-2$ times and the generators of the other regulus twice, and therefore it has ∞^1 $(n-2)$-secant lines. Through a general point P pass ∞^1 such curves all having an $(n-2)$-secant line in common and the 3-spaces containing them all pass through this common $(n-2)$-secant line. These 3-spaces through P form a quadric hypersurface V_4^2 of S_5.

Now project this F^{n+2} from P upon an S_4 and the projection, F^{n+1}, has an improper multiple point of order $n-3$ which is to be regarded as the union of $(n-3)(n-4)/2$ improper double points and which is the intersection of the common $(n-2)$-secant line of the ∞^1 3-space n-ic curves on F^{n+2} through P. On F^{n+1} are ∞^1 plane curves of order $n-1$ all having an $(n-3)$-fold point at the improper $(n-3)$-fold point. These ∞^1 curves are the projections of the 3-space n-ic curves through P. Now in the plane f of representation there are $3n-5$ simple base points $B_1, \cdots, B_{3n-5}$ besides the $(n-2)$-fold base point A. The pencil of $(n-1)$-ic curves through A $n-3$ times and B_i once yields the ∞^1-system of plane $(n-1)$-ic curves on F^{n+1} with the same $(n-3)$-fold point at the improper $(n-3)$-fold point of the surface. The planes of these curves generate a V_3^2 which is the intersection of the V_4^2, mentioned in the preceding paragraph, and S_4. Note that the $n-3$ base points of the pencil of $(n-1)$-ic curves of f, distinct from A and B_i, are the images of the improper $(n-3)$-fold point of F^{n+1}.

A general section of F^{n+1} by an S_3 is a curve having ∞^1 $(n-1)$-secant lines lying on a quadric surface which is the section of V_3^2 by S_3. This curve meets the generators of one regulus of this surface in $n-1$ points and those of the other in two points. It is the partial intersection of the quadric surface and another surface, of order $2n-2$, having $n-3$ lines in common.

Now of the ∞^1 $(n-1)$-ic curves on F^{n+1} one, say K^{n-1}, passes through a general point Q. Projecting F^{n+1} from Q upon S_3, we obtain an F^n with an $(n-2)$-fold line l which is the projection of K^{n-1}. The curve K^{n-1} having an $(n-3)$-fold point is of genus 3 and class $4n-10$. The point Q being on K^{n-1}, the number of tangent lines from Q to K^{n-1} is therefore $4n-12$. The projections of the points of contact are pinch points on F^n. Hence F^n has $4n-12$ pinch points on the $(n-2)$-fold line l.

If we project F^{n+1} from its improper $(n-3)$-fold point upon S_3, we obtain for projection a quartic surface composed of two coincident quadric surfaces. This double quadric surface is the intersection of the V_3^2 already mentioned and S_3.

It is of interest to note that the F^{4n-4} in S_{3n-1} may be represented upon f by an ∞^{3n-1}-system of $(n+N)$-ic curves having one $(n+N-2)$-fold base point and N double base points for all values of $N \geqq 0$. A general projection of F^{4n-4} upon an S_4 has $8n^2-31n+31$ improper double points all lying in a plane π. This plane π contains a curve K^{4n-6}, of order $4n-6$, of the projected surface and this is of genus $n-3$ and has the $8n^2-31n+31$ improper double points for nodes. If we project the projected surface upon an S_3, we have an F'^{4n-4} having a double curve of order $8n^2-23n+17$ with $16n-28$ pinch points. If the center of projection is in π, the double curve degenerates into a $(4n-6)$-fold line and $3n-4$ double lines. Since the class of K^{4n-6} is $10n-20$, there are on the $(4n-6)$-fold line $10n-20$ pinch points. The remaining $6n-8$ pinch points are on the $3n-4$ double lines, 2 on each.

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SPINORS AND TENSORS

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It is well known that there are two kinds of quantities connected with the representations of a group of rotations—the *tensors* and the *spinors*.* Since the advent of the relativity theory we had been led to believe, in the words of O. Veblen,† "that any physical phenomena could be described by means of tensors." But then came the Dirac equations of the electron which give an example of a situation described in terms of spinors. Does it mean that we have to change the belief expressed above? It does not follow. All that has happened is that we have a phenomenon not described in terms of tensors; that does not mean that it cannot be so described. That it might be possible to describe every situation given in spinors also in tensors is suggested by the fact that there exist algebraic relations between spinors and tensors; it may be possible to eliminate the spinors from a sufficient number of these algebraic relations and the given spinor differential equations, and obtain in this way an equivalent description in tensors. The discussion of the general case should not be very difficult, but it seemed that a simple special case should be worked out first, and that is why I suggested to Gordon Fuller the problem which he discusses in his article.‡ The problem there is treated without

* Compare, for example, R. Brauer and H. Weyl, American Journal of Mathematics, vol. 40 (1935), p. 425.

† Proceedings of the National Academy of Sciences, vol. 24 (1934), p. 282.

‡ This issue of this Bulletin, vol. 42 (1936), p. 107.