ON SOME EXTREMAL PROPERTIES OF TRIGONOMETRIC POLYNOMIALS WITH REAL ROOTS

BY J. GERONIMUS

1. *Introduction.* L. Fejér [1],† O. Szász [2], [3], [4] and E. v. Egerváry [5] have found many interesting extremal properties of non-negative trigonometric polynomials. In particular, Szász [3] has found that for every non-negative trigonometric polynomial of order $\leqq n$ with real coefficients,

$$(1)\quad G_n(\theta) = 1 + \Re\sum_{k=1}^{n} \bar{\gamma}_k e^{ik\theta}, \quad (\gamma_k = \alpha_k + i\beta_k;\ k = 1, 2, \cdots, n),$$

the inequality

$$(2)\qquad |\gamma_k| \leqq 2\cos\frac{\pi}{\left[\dfrac{n}{k}\right] + 2}, \qquad (k = 1, 2, \cdots, n),$$

is valid.‡

The object of this note is to find the *minimum* of the modulus of the first coefficient γ_n, supposing that all roots of $G_n(\theta)$ are real. The first problem of this kind has been considered by Blumenthal [6]; we shall return in §4 to his problem and its generalization.

2. *Equality of Roots of $G_n^*(\theta)$ for Problem* 1. Consider the following problem.

PROBLEM 1. *Find the minimum of the modulus of the first coefficient γ_n of a non-negative trigonometric polynomial*

$$G_n(\theta) = 1 + \Re\sum_{k=1}^{n} \bar{\gamma}_k e^{ik\theta}$$

of order n with real roots.

† Numbers in brackets refer to the Bibliography at the end.

‡ $\Re z$ means real part of z; $[a]$ means the greatest integer $\leqq a$.

In order to solve this problem we shall prove the following simple lemma.

LEMMA 1. *All roots of the polynomial $G_n^*(\theta)$ for which the minimum in Problem 1 is attained must be equal.*

Consider a non-negative trigonometric polynomial†

$$(3)\quad G_n(\theta) = 4\sin^2\frac{\theta-\theta_1}{2}\sin^2\frac{\theta-\theta_2}{2}F_{n-2}(\theta) = \Re\sum_{k=0}^{n}\bar{\gamma}_k e^{ik\theta},$$

where $F_{n-2}(\theta)$ is a non-negative trigonometric polynomial of order $n-2$ with real roots,

$$(4)\quad F_{n-2}(\theta) = \Re\sum_{k=0}^{n-2}\bar{\gamma}_k^* e^{ik\theta} = \sum_{k=0}^{n-2}|\gamma_k^*|\cos(k\theta-\alpha_k),$$

where $\alpha_k = \arg\gamma_k^*$, $(k=0, 1, 2, \cdots, n-2)$, and $\alpha_0 = 0$.

On putting $\alpha = (\theta_1+\theta_2)/2$, $\delta = (\theta_1-\theta_2)/2$, we see easily that

$$(5)\quad \begin{aligned}\gamma_0 &= \gamma_0^*\left(1+\frac{1}{2}\cos 2\delta\right) - |\gamma_1^*|\cos(\alpha_1-\alpha)\cos\delta\\ &\quad + \frac{1}{4}|\gamma_2^*|\cos(\alpha_2-2\alpha); \qquad |\gamma_n| = \frac{1}{4}|\gamma_{n-2}^*|.\end{aligned}$$

We see that $|\gamma_n|$ does not depend on α, nor on δ; on the other hand γ_0 is maximal for $\delta=0$ if $\cos(\alpha_1-\alpha)\leqq 0$, or for $\delta=\pi$ if $\cos(\alpha_1-\alpha)\geqq 0$. In both cases the minimal value of $|\gamma_n|$ under condition $\gamma_0=1$ corresponds to $\delta=0$ or $\delta=\pi$; therefore θ_1 and θ_2 coincide.‡

3. *Polynomials for which γ_n has Extremal Values.* It follows from this lemma that $G_n^*(\theta)$ is

$$(6)\quad G_n^*(\theta) = C[1+\cos(\theta+\alpha)]^n,$$

α being an arbitrary real argument; it may be written thus:§

$$(7)\quad G_n^*(\theta) = \frac{C}{2^{n-1}}\left\{\frac{1}{2}C_{2n,n} + \sum_{k=1}^{n}C_{2n,n-k}\cos k(\theta+\alpha)\right\}.$$

† It is clear that all real roots of a non-negative trigonometric polynomial are of even multiplicity.

‡ θ_1 and $\theta_1+2\pi$ are not considered as different.

§ See [6], p. 392.

For this polynomial we have

$$\gamma_0 = \frac{1}{2^n} C(C_{2n,n}); \qquad |\gamma_n| = \frac{1}{2^{n-1}} C,$$

whence we find the ratio

$$\frac{|\gamma_n|}{\gamma_0} = \frac{2}{C_{2n,n}}. \tag{8}$$

We have proved the following theorem.

THEOREM 1. *If $G_n(\theta)$ is a non-negative trigonometric polynomial,*

$$G_n(\theta) = 1 + \Re \sum_{k=1}^{n} \bar{\gamma}_k e^{ik\theta},$$

of order n with real roots, then

$$\frac{2}{C_{2n,n}} \leqq |\gamma_n| \leqq 1; \tag{9}$$

the maximum is attained for the polynomial†

$$G_{\max}(\theta) = 1 + \cos n(\theta + \alpha), \tag{10}$$

and the minimum for the polynomial

$$G_{\min}(\theta) = \frac{2^n}{C_{2n,n}} \{1 + \cos(\theta + \alpha)\}^n, \tag{11}$$

α being an arbitrary real argument.

4. *The Generalized Extremal Problem.* Consider now the following extremal problem.

PROBLEM 2. *Find the minimum of the ratio*

$$\frac{A_m^2 + B_m^2}{\lambda A_0^2 + \sum_{k=1}^{m} (A_k^2 + B_k^2)}, \tag{12}$$

† See [1], [2].

where

$$g_m(\theta) = A_0 + A_1 \cos\theta + B_1 \sin\theta + \cdots + A_m \cos m\theta + B_m \sin m\theta$$

is a trigonometric polynomial of order m with real roots, and λ *is an arbitrary non-negative number.*

The above mentioned problem of Blumenthal [6] corresponds to $\lambda = 1$. It is easy to see that for $\lambda = 2$ Problem 2 is a particular case of the Problem 1. Indeed we see that

$$(13) \qquad g_m^2(\theta) = G_n(\theta) = \mathcal{R}\sum_{k=0}^{n} \bar{\gamma}_k e^{ik\theta}$$

is a non-negative trigonometric polynomial of order $n = 2m$, while

$$(14) \quad \gamma_0 = A_0^2 + \frac{1}{2}\sum_{k=1}^{m}(A_k^2 + B_k^2), \qquad |\gamma_n| = \frac{A_m^2 + B_m^2}{2};$$

therefore we have for $\lambda = 2$

$$(15) \qquad \frac{A_m^2 + B_m^2}{2A_0^2 + \sum_{k=1}^{m}(A_k^2 + B_k^2)} \geqq \frac{2}{C_{2n,n}} = \frac{2}{C_{4m,2m}}.$$

To solve our problem for all $\lambda \geqq 0$ we shall put it in the following form.

PROBLEM $2'$. *Find the maximum of the expression*

$$(16) \quad L(G) = \frac{1}{2\pi}\int_0^{2\pi} G_n(\theta)d\theta + \epsilon\left\{\frac{1}{2\pi}\int_0^{2\pi} [G_n(\theta)]^{1/2} d\theta\right\}^2, \qquad (\epsilon \geqq -1),$$

where $G_n(\theta)$ *is a non-negative trigonometric polynomial*

$$G_n(\theta) = \mathcal{R}\sum_{k=0}^{n} \bar{\gamma}_k e^{ik\theta}, \qquad (|\gamma_n| = 1),$$

of order $n = 2m$ *with real roots.*

5. *Equality of Roots of* $G_n^*(\theta)$ *for Problem* $2'$. We shall prove the following lemma.

LEMMA 2. *All roots of the polynomial $G_n^*(\theta)$ for which the maximum in Problem 2' is attained must be equal.*

Put

$$(17)\qquad \begin{aligned}[G_n(\theta)]^{1/2} &= 2\sin\frac{\theta-\theta_1}{2}\sin\frac{\theta-\theta_2}{2}F_{m-1}(\theta)\\ &= [\cos\delta - \cos(\theta-\alpha)]F_{m-1}(\theta),\end{aligned}$$

where $\alpha=(\theta_1+\theta_2)/2$, $\delta=(\theta_1-\theta_2)/2$, and $F_{m-1}(\theta)$ is a non-negative trigonometric polynomial of order $m-1$,

$$(18)\qquad F_{m-1}(\theta) = \mathcal{R}\sum_{k=0}^{m-1}\bar{c}_k e^{ik\theta},$$

with real roots. Thus we get

$$(19)\qquad \frac{1}{2\pi}\int_0^{2\pi}[G_n(\theta)]^{1/2}d\theta = c_0\cos\delta - \frac{1}{2}|c_1|\cos(\beta_1-\alpha),$$

where $\beta_k=\arg c_k$, $(k=0, 1, \cdots, m-1)$, and $\beta_0=0$. Further let

$$(20)\qquad F_{m-1}^2(\theta) = \mathcal{R}\sum_{k=0}^{2m-2}\bar{c}_k^* e^{ik\theta};$$

then we obtain

$$(21)\qquad \begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}G_n(\theta)d\theta &= c_0^*\left(1+\frac{1}{2}\cos 2\delta\right)\\ &\quad - |c_1^*|\cos(\beta_1^*-\alpha)\cos\delta + \frac{1}{4}|c_2^*|\cos(\beta_2^*-2\alpha),\end{aligned}$$

where $\beta_k^*=\arg c_k^*$, $(k=0, 1, \cdots, 2m-2)$, and $\beta_0^*=0$. Using (19) and (21) we have

$$(22)\qquad L(G) = A\cos 2\delta + B\cos\delta + C,$$

where

$$(23)\qquad \begin{aligned}A &= \frac{1}{2}(c_0^*+\epsilon c_0^2),\\ B &= -|c_1^*|\cos(\beta_1^*-\alpha) - \epsilon c_0|c_1|\cos(\beta_1-\alpha),\\ C &= \frac{1}{4}|c_2^*|\cos(\beta_2^*-2\alpha)\\ &\quad + \frac{1}{4}\epsilon|c_1|^2\cos^2(\beta_1-\alpha) + c_0^* + \frac{1}{2}\epsilon c_0^2.\end{aligned}$$

It is important to point out that $|\gamma_n| = (1/4)|c^*_{2m-2}|$ does not depend on α, nor on δ. Since we have

$$(24) \qquad c_0^* = c_0^2 + \frac{1}{2}\sum_{k=1}^{m-1} |c_k|^2,$$

it is clear that for $\epsilon \geqq -1$ we have

$$(25) \qquad A = \frac{1}{2}(1+\epsilon)c_0^2 + \frac{1}{4}\sum_{k=1}^{m-1} |c_k|^2 > 0.$$

Therefore $L(G_n)$ is maximal for $\delta = 0$ if $B \geqq 0$, and for $\delta = \pi$ if $B \leqq 0$; in both cases θ_1 and θ_2 coincide, which proves our lemma.

6. *Polynomials having the Extremal Property.* We see that the polynomial $G_n^*(\theta)$ is

$$(26) \qquad \begin{aligned} G_n^*(\theta) &= 2^{n-1}[1 + \cos(\theta+\alpha)]^n \\ &= \frac{1}{2}C_{2n,n} + \sum_{k=1}^{n} C_{2n,n-k}\cos k(\theta+\alpha), \end{aligned}$$

and we have for it

$$(27) \qquad L(G_n^*) = \frac{1}{2}(C_{2n,n} + \epsilon(C_{n,n/2})^2).$$

Thus we have proved the following theorem.

THEOREM 2. *If $G_n(\theta)$ is a non-negative trigonometric polynomial of order $n = 2m$,*

$$G_n(\theta) = \Re\sum_{k=0}^{n} \bar{\gamma}_k e^{ik\theta}, \qquad |\gamma_n| = 1,$$

with real roots, then

$$(28) \qquad \begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} G_n(\theta)d\theta + \epsilon\left\{\frac{1}{2\pi}\int_0^{2\pi}[G_n(\theta)]^{1/2}d\theta\right\}^2 \\ &\qquad \leqq \frac{1}{2}(C_{2n,n} + \epsilon(C_{n,n/2})^2), \qquad (\epsilon \geqq -1); \end{aligned}$$

the maximum is attained for the polynomial

$$G_n^*(\theta) = 2^{n-1}\{1 + \cos(\theta + \alpha)\}^n, \tag{29}$$

α being an arbitrary real argument.

This result may also be stated as the following theorem.

THEOREM 2′. *If $g_m(\theta)$ is a trigonometric polynomial of order m,*

$$g_m(\theta) = A_0 + A_1 \cos\theta + B_1 \sin\theta + \cdots + A_m \cos m\theta + B_m \sin m\theta,$$

with real roots, then

$$\frac{A_m^2 + B_m^2}{\lambda A_0^2 + \sum_{k=1}^{m} (A_k^2 + B_k^2)} \geqq \frac{2}{C_{4m,2m} + \frac{\lambda - 2}{2}(C_{2m,m})^2}, \quad (\lambda \geqq 0); \tag{30}$$

this minimum is attained for the polynomial

$$g_m^*(\theta) = C\{1 + \cos(\theta + \alpha)\}^m. \tag{31}$$

BIBLIOGRAPHY

1. L. Fejér, *Über trigonometrische Polynome*, Journal für Mathematik, vol. 146 (1915), pp. 53–82.

2. O. Szász, *Über nichtnegative trigonometrische Polynome*, Sitzungsberichte der Bayerischen Akademie der Wissenschaften, (1917), pp. 307–320.

3. O. Szász, *Elementare Extremalprobleme über nichtnegative trigonometrische Polynome*, ibid., (1927), pp. 185–196.

4. O. Szász, *Über harmonische Funktionen und L-Formen*, Mathematische Zeitschrift, vol. 1 (1918), pp. 149–162.

5. E. v. Egerváry und O. Szász, *Einige Extremalprobleme im Bereiche der trigonometrischen Polynome*, ibid., vol. 27 (1927), pp. 641–652.

6. O. Blumenthal, *Einige Minimums-Sätze über trigonometrische und rationale Polynome*, Mathematische Annalen, vol. 77 (1916), pp. 390–403.

MATHEMATICAL INSTITUTE,
KHARKOW, U.S.S.R.