GENERAL SOLUTION OF THE PROBLEM OF ELASTOSTATICS OF AN n-DIMENSIONAL HOMOGENEOUS ISOTROPIC SOLID IN AN n-DIMENSIONAL SPACE

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1. *Introduction.* Dealing with the important case of a three-dimensional solid subject to constant body forces (such as gravity) B. Galerkin* expressed the stresses and the displacements in terms of three functions, governed by the fourth-order equation $\Delta\Delta f=$const., and mutually independent except through the boundary conditions. He has demonstrated the fruitfulness of his method in later papers.†

It is profitable to interpret Galerkin's three functions as components of a vector. Simplicity is gained and significance is added by doing this. It is proposed to call this vector *the Galerkin vector.* Its nature is such that only a slight amount of complexity is added in the general derivations by considering an n-dimensional space.

2. *Notation.* Let the following notation be used.

$i_1, i_2, \cdots, i_m, \cdots, i_p, \cdots, i_n=$ unit vectors in n directions perpendicular to one another; $m\neq p$.

$\boldsymbol{R}=i_1x_1+i_2x_2+\cdots+i_nx_n=$ radius vector drawn from the origin to any point; the point is called point $\boldsymbol{R}$.

$\boldsymbol{\rho}=i_1\xi_1+i_2\xi_2+\cdots+i_n\xi_n=$ displacement $=$ increment of $\boldsymbol{R}$. The point $\boldsymbol{R}$ moves to the position $\boldsymbol{R}+\boldsymbol{\rho}$; $\boldsymbol{\rho}$ is assumed small.

$\boldsymbol{P}=i_1P_1+i_2P_2+\cdots+i_nP_n=$ force.

$\boldsymbol{K}=i_1K_1+i_2K_2+\cdots+i_nK_n=$ body force which is distrib-

* B. Galerkin, *Contribution à la solution générale du problème de la théorie de l'élasticité dans le cas de trois dimensions*, Comptes Rendus, vol. 190 (1930), p. 1047; *Contribution à l'investigation des tensions et des déformations d'un corps élastique isotrope* (in Russian), Comptes Rendus de l'Académie des Sciences de l'URSS, (1930), p. 353.

† Comptes Rendus, vol. 193 (1931), p. 568; vol. 194 (1932), p. 1440; vol. 195 (1932), p. 858; and papers in Russian: Comptes Rendus de l'Académie des Sciences de l'URSS, (1931), p. 273 and p. 281; Messenger of Mechanics and Applied Mathematics, Leningrad, vol. 1 (1931), p. 49; Transactions of the Scientific Research Institute of Hydrotechnics, vol. 10 (1933), p. 5.

uted through the solid, per unit of magnitude of the region considered; of the dimension force times distance $^{-n}$; a function of $\boldsymbol{R}$.

S_m = section defined by x_m = const. The section is called a front face if the part of the solid dealt with lies on the side of smaller values of x_m; and a back face if the part dealt with lies on the side of greater values of x_m.

$\mathbf{s}_m$ = vector representing internal force or stress on a small region of the front face S_m at point $\boldsymbol{R}$, per unit of magnitude of this region; of the dimension force times distance^{-n+1}; and $-\mathbf{s}_m$ = stress on the same region of the back face S_m.

σ_m = component of $\mathbf{s}_m$ in the direction $\boldsymbol{i}_m$; normal stress in the direction of $\boldsymbol{i}_m$.

τ_{mp} = component of $\mathbf{s}_m$ in the direction of $\boldsymbol{i}_p$, $(p \neq m)$; shearing stress on the front face S_m in the direction $\boldsymbol{i}_p$.

$\Theta = \sigma_1 + \sigma_2 + \cdots + \sigma_n$ = bulk stress.

E = Young's modulus of elasticity (equation (6)).

G = modulus of elasticity in shear (equation (7)).

μ = Poisson's ratio of lateral contraction (equation (6)).

$\boldsymbol{F} = \boldsymbol{i}_1 X_1 + \boldsymbol{i}_2 X_2 + \cdots + \boldsymbol{i}_n X_n$ = Galerkin vector.

$$\nabla = \boldsymbol{i}_1 \frac{\partial}{\partial x_1} + \boldsymbol{i}_2 \frac{\partial}{\partial x_2} + \cdots + \boldsymbol{i}_n \frac{\partial}{\partial x_n}.$$

$$\operatorname{div} \boldsymbol{F} = \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \cdots + \frac{\partial X_n}{\partial x_n}.$$

$$\Delta = \operatorname{div} \nabla = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

3. *Equilibrium.* A set of forces $\boldsymbol{P}$ applied at the points $\boldsymbol{R}$ is in equilibrium when the n conditions

$$\sum P_m = 0, \tag{1}$$

and the $n(n-1)/2$ conditions

$$\sum P_m(x_p - C_p) = \sum P_p(x_m - C_m) \tag{2}$$

are satisfied, each sum including all the force components in the single direction of either $\boldsymbol{i}_m$ or $\boldsymbol{i}_p$, C_m and C_p being constants which are interpreted as coordinates in the directions of $\boldsymbol{i}_m$ and $\boldsymbol{i}_p$ of an arbitrary fixed point C.

Consider an element of the solid bounded by back faces $S_1, S_2, \cdots, S_n$ intersecting at the point $\boldsymbol{R}$, and by front faces

$S_1, S_2, \cdots, S_n$ intersecting at the point $\boldsymbol{R}+d\boldsymbol{R}$. With all the forces included which act on this element, and with C at the center of the element, equation (2) leads to the $n(n-1)/2$ conditions

$$\tau_{mp} = \tau_{pm}. \tag{3}$$

Equation (1) leads to the n conditions

$$\frac{\partial \sigma_m}{\partial x_m} + \sum_{\substack{1,2,\cdots,n \\ \text{excl. } m}}^{p} \frac{\partial \tau_{pm}}{\partial x_p} + K_m = 0. \tag{4}$$

Because of equation (3), equation (4) may be rewritten in the form

$$\operatorname{div} s_m + K_m = 0. \tag{5}$$

4. *Elasticity*. Hooke's law of stresses and deformations is stated in the following form. The strain in the direction of i_m is

$$\frac{\partial \xi_m}{\partial x_m} = \frac{1+\mu}{E}\sigma_m - \frac{\mu}{E}\Theta, \tag{6}$$

and the detrusion in the directions of i_m and i_p is

$$\frac{\partial \xi_m}{\partial x_p} + \frac{\partial \xi_p}{\partial x_m} = \frac{\tau_{mp}}{G}. \tag{7}$$

With the constants E and μ given, only one value of G leads to isotropy. The value

$$G = \frac{E}{2(1+\mu)} \tag{8}$$

represents isotropy in the cases $n=2$ and $n=3$, and is assigned to G here. That this value actually represents isotropy for any value of n is concluded from the form of equation (13), which is derived from equations (5) to (8). Equation (6) gives

$$\operatorname{div} \boldsymbol{\varrho} = \frac{1-(n-1)\mu}{E}\Theta, \tag{9}$$

which, with (8), permits the rewriting of (6) in the form

$$\sigma_m = 2G\left(\frac{\partial \xi_m}{\partial x_m} + \frac{\mu}{1-(n-1)\mu}\operatorname{div} \boldsymbol{\varrho}\right). \tag{10}$$

Equations (7) and (10) lead to the following formula for the resultant stress on the front face S_m:

$$s_m = G\left(\nabla \xi_m + \frac{\partial \mathfrak{g}}{\partial x_m} \right) + i_m \frac{2G\mu}{1 - (n - 1)\mu} \operatorname{div} \mathfrak{g}. \tag{11}$$

Substitution from equation (11) in equation (5) gives

$$\Delta \xi_m + \frac{1 - (n - 3)\mu}{1 - (n - 1)\mu} \frac{\partial}{\partial x_m} \operatorname{div} \mathfrak{g} + \frac{K_m}{G} = 0, \tag{12}$$

or, in vector form,

$$\left(\Delta + \frac{1 - (n - 3)\mu}{1 - (n - 1)\mu} \nabla \operatorname{div} \right) \mathfrak{g} + \frac{K}{G} = 0. \tag{13}$$

Isotropy is attained because equation (13) is independent of the orientation of the axes of coordinates.

5. *The Galerkin Vector.* The general solution of equation (13) is the general solution of the problem. A difficulty arises from the interdependence of the components. This difficulty is overcome by introducing the Galerkin vector $\boldsymbol{F}$.

The displacement is expressed as

$$\mathfrak{g} = \frac{1}{2G} (c\Delta - \nabla \operatorname{div}) \boldsymbol{F} \tag{14}$$

in which c is a constant yet to be selected. By substituting $\mathfrak{g}$ from equation (14) in equation (13) and noting that

$$\Delta \nabla \operatorname{div} = \nabla \operatorname{div} \Delta = \nabla \operatorname{div} \nabla \operatorname{div}, \tag{15}$$

it is found that the terms containing the combined operators shown in equations (15) disappear when

$$c = \frac{2(1 - (n - 2)\mu)}{1 - (n - 3)\mu}. \tag{16}$$

Then equations (14) and (13) become

$$\mathfrak{g} = \frac{1}{2G} \left[\frac{2(1 - (n - 2)\mu)}{1 - (n - 3)\mu} \Delta - \nabla \operatorname{div} \right] \boldsymbol{F}, \tag{17}$$

$$\Delta^2 \boldsymbol{F} = - \frac{1 - (n - 3)\mu}{1 - (n - 2)\mu} \boldsymbol{K}. \tag{18}$$

The general solution of equation (18) defines the general solution for $\boldsymbol{\varrho}$ through equation (17).

6. *Stresses.* The vector $\boldsymbol{F}$ and its components $X_1, X_2, \cdots, X_n$ lend themselves to expressions for the stresses. Equation (17) gives

$$(19)\qquad \xi_m = \frac{1}{2G}\left[\frac{2(1-(n-2)\mu)}{1-(n-3)\mu}\Delta X_m - \frac{\partial}{\partial x_m}\operatorname{div}\boldsymbol{F}\right],$$

and

$$(20)\qquad \operatorname{div}\boldsymbol{\varrho} = \frac{1-(n-1)\mu}{2(1-(n-3)\mu)G}\Delta\operatorname{div}\boldsymbol{F}.$$

Substitution from equations (19) and (20) in equations (10), (7), (9), and (11) leads to the formulas

$$(21)\qquad \begin{aligned}\sigma_m &= \frac{2(1-(n-2)\mu)}{1-(n-3)\mu}\frac{\partial\Delta X_m}{\partial x_m}\\ &\quad + \left[\frac{\mu}{1-(n-3)\mu}\Delta - \frac{\partial^2}{\partial x_m^2}\right]\operatorname{div}\boldsymbol{F},\end{aligned}$$

$$(22)\qquad \tau_{mp} = \frac{1-(n-2)\mu}{1-(n-3)\mu}\left[\frac{\partial\Delta X_m}{\partial x_p} + \frac{\partial\Delta X_p}{\partial x_m}\right] - \frac{\partial^2}{\partial x_m\partial x_p}\operatorname{div}\boldsymbol{F},$$

$$(23)\qquad \Theta = \frac{1+\mu}{1-(n-3)\mu}\Delta\operatorname{div}\boldsymbol{F},$$

and

$$(24)\qquad \begin{aligned}\boldsymbol{s}_m &= \frac{1-(n-2)\mu}{1-(n-3)\mu}\left[\nabla\Delta X_m + \frac{\partial}{\partial x_m}\Delta\boldsymbol{F}\right]\\ &\quad + \left[\frac{\boldsymbol{i}_m\mu}{1-(n-3)\mu}\Delta - \frac{\partial}{\partial x_m}\nabla\right]\operatorname{div}\boldsymbol{F}.\end{aligned}$$

The form of equations (17) and (18) shows that $\boldsymbol{F}$ is independent of the orientation of the axes of coordinates. It follows that equations (19) to (24) continue to apply with the same $\boldsymbol{F}$ after a re-orientation of the axes of coordinates.

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