A REMARK ON METHOD IN TRANSFINITE ALGEBRA†

BY MAX ZORN‡

The theorems of Steinitz concerning algebraic closure and the degree of transcendence are barred, from the algebraic point of view, by the well-ordering theorem and its theory. We wish to show how, by introducing a certain axiom on sets of sets instead of the well-ordering theorem, one is enabled to make the proofs shorter and more algebraic. The proofs will be given in terms of the non-axiomatic standpoint of set theory.

DEFINITION 1. A set $\mathfrak{B}=\{B\}$ of sets B is called a *chain*, if for every two sets B_1, B_2, either $B_1 \supset B_2$ or $B_2 \supset B_1$.

DEFINITION 2. A set $\mathfrak{A}$ of sets A is said to be closed (right-closed), if it contains the union $\sum_{\mathfrak{B}\ni B} B$ of every chain $\mathfrak{B}$ contained in $\mathfrak{A}$.

Then our *maximum principle* is expressible in the following form.

(MP). *In a closed set $\mathfrak{A}$ of sets A there exists at least one, A^*, not contained as a proper subset in any other $A \in \mathfrak{A}$.*

APPLICATIONS. I. Let $\mathfrak{R}$ be a ring with a unity element 1; let $\mathfrak{A}$ be the set of all ideals $\mathfrak{a}$ (i) not containing 1 as an element, (ii) containing a certain ideal $\mathfrak{r} \neq \mathfrak{R}$. The set $\mathfrak{A}$ is obviously closed; the maximum principle implies the existence of a maximal ideal $\mathfrak{p}$ with $\mathfrak{r} \subseteq \mathfrak{p} \subset \mathfrak{R} \neq \mathfrak{p}$; this ideal is a prime ideal and the residue class ring $\mathfrak{R}/\mathfrak{p}$ is a *field*.

II. If k is a real field, that is, a field such that no sum of squares vanishes unless all the squares vanish, and K is an arbitrary algebraical extension, then the set of all real fields between k and K is closed, so the MP assures the existence of a *maximal* real field between k and K. In particular, if K is algebraically closed, we obtain a real closure of k.

III. Let K be an arbitrary field extension of k. A set of K-elements $\{a\}$ is said to be algebraically (respectively, linearly) independent, if no finite subset $a_1, a_2, a_3, \cdots, a_n$ satisfies an

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‡ Sterling Fellow.

algebraic (linear) relation with coefficients in k (not all vanishing). The set of all these independent sets is closed; a maximal independent system, which exists as a consequence of the MP, is a basis for K/k with respect to algebraic (linear) dependence. (The theorems concerning dependence are consequences of the dependence axioms† only.)

We shall first prove the existence of an algebraic, algebraically closed field extension. Indeed, we shall construct for each set of polynomials $\{p(x)\}$ with coefficients in k a minimal decomposition field. (Let p have the highest coefficient 1 and degree n_p.)

We form the field k' of all rational functions in x over k. Over k' we consider the domain of all polynomials in the following set of indeterminates: $y_{p,i}$, the first index p ranging over all polynomials of our set, the second over the integers from 1 to n_p.

In this domain of integrity let us take the ideal generated by the polynomials

$$\{p(x) - (x - y_{p,1})(x - y_{p,2}) \cdots (x - y_{p,n_p})\}.$$

Suppose that this ideal does not contain the number 1. Then we have a prime ideal containing it, and the residue classes of the indeterminates y generate a minimal field over k, in which all polynomials $p(x)$ are completely decomposed.

If now this ideal should contain the unity element, we should have a representation of 1 as a linear combination of a finite set of differences $p(x)-(x-y_{p,1}) \cdots$ with coefficients rational in x and integral in the y's. This relation in indeterminates y gives the contradiction 1 equal to 0, if we substitute for the indeterminates the roots of the *finite* set of polynomials p occurring in the relation; these roots exist in accordance with *elementary* theorems in a suitable field extension.

Let K_1, K_2 be two algebraic extensions of k, which are minimal decomposition fields for the set of polynomials $\{p(x)\}$ over k. We consider the partial realizations of the uniqueness theorem, that is, the isomorphisms between fields K_1^2 and K_2^2, which are the identity in the common subfield k, K_i^2 lying in K_i. The function tables of these isomorphisms, considered as sets of pairs, form a *closed* system (of sets of pairs); therefore a maximal one must exist in accordance with MP.

† Van der Waerden, *Moderne Algebra*, vol. 1, p. 204.

This maximal isomorphism must range over the entire fields K_i. For let K_i^0 be the fields connected by the maximal isomorphism. Take an arbitrary polynomial $p(x)$ in the set $\{p\}$. The minimal decomposition fields for this polynomial $p(x)$ over K_i^0, generated by adjoining respectively the roots in K_i, are *isomorphic* under an isomorphism which continues our maximal isomorphism between K_1^0 and K_2^0. (As in the case of the existence these proofs are based on the corresponding *finite* theorems; precisely as in the proof dealing with the degree of transcendence.) Hence the decomposition fields reduce to K_i^0; K_i^0 decomposes all polynomials of the set and is therefore identical with K_i; our maximal isomorphism yields the uniqueness of the minimal decomposition field, in particular, the uniqueness of algebraic closure.

Finally we prove the existence of the degree of transcendence. Let $\{x\}$ and $\{y\}$ be two bases of K/k. Once more we consider the partial realizations of our theorem. Namely, we take a (1, 1) correspondence between subsets $\{x\}'$ and $\{y\}'$ of $\{x\}$ and $\{y\}$, where $\{x\}'$ and $\{y\}'$ are *dependent.*

The function tables of these correspondences, considered as sets of pairs, form a *closed* system. We shall prove by the corresponding *finite* theorem concerning degree of transcendence and the axiom of *choice* that a maximal correspondence of this kind which exists in accordance to MP must range over the entire bases $\{x\}$ and $\{y\}$. Take a correspondence with $\{x\}'$ different from $\{x\}$. Take a finite set of x's not contained in $\{x\}'$. This will depend on $\{y\}'$ and a certain finite set of y's which is not contained in $\{y\}'$. This finite set of y's depends on $\{x\}'$, the arbitrarily chosen finite set of x's and a certain other finite set of new x's; proceeding in zigzag we obtain (choice axiom) two sequences of new x's and y's; we may proceed similarly, in case of a finite number (finite theorem) or an infinite (denumerable) set. It is not necessary to show the utility of the MP in further algebraic or analytic examples.

In another paper I shall discuss the relations between MP, the axiom of choice, and the well-ordering theorem. I shall show that they are equivalent if the axiom yielding the set of all subsets of a set is available. It seems that without this axiom the MP is weaker than the well-ordering theorem; the situation may be studied by means of other axioms.

In the present paper all the axioms concerning sets are used;

in the case of algebraic closure there is a stronger result of Hollkott (Hamburg): the axiom of choice is sufficient for the existence and uniqueness of algebraic closure.

An essential simplification is made possible for Baer's† theory of the degree of algebraic extensions. I plan to show elsewhere how the generalized continuum hypothesis may be avoided.

YALE UNIVERSITY

CONCERNING TWO INTERNAL PROPERTIES OF PLANE CONTINUA‡

BY R. E. BASYE

Theorem 1 below was suggested to me by R. L. Moore. Theorem 2 is an extension of Kuratowski's result§ that if three compact plane continua have a point in common and their sum separates a point A from a point B in the plane, then there exists a pair of these continua whose sum separates A from B in the plane. Another extension of this result along combinatorial lines has been given|| by Čech.

THEOREM 1. *Let H and K be two mutually exclusive and closed subsets of a compact continuum M which lies in the plane. If for each pair of points A and B in H and K, respectively, there exists a finite collection Γ_{AB} of continua in M such that Γ_{AB}^* separates A from B in M, then there exists a finite collection Γ of continua in M such that Γ^* separates H from K in M.*

Let $\epsilon_1, \epsilon_2, \cdots$, be a sequence of positive numbers converging monotonically to zero, with ϵ_1 less than half the distance from H to K. For each i let D_H^i be a domain containing H such that (1) the boundary β_H^i of D_H^i is the sum of a finite number of mutually exclusive simple closed curves, and (2) each point of

† *Eine Anwendung der Kontinuumhypothese in der Algebra.* Journal für Mathematik, vol. 162.

‡ Presented to the Society, April 6, 1935, under a somewhat different title.

§ Kuratowski, *Théorème sur trois continus,* Monatshefte für Mathematik und Physik, vol. 36 (1929), pp. 77–80.

|| E. Čech, *Trois théorèmes sur l'homologie,* Publications de la Faculté des Sciences de L'Université Masaryk, No. 144, 1931, pp. 1–21.