FURTHER NON-INVOLUTORIAL CREMONA SPACE TRANSFORMATIONS CONTAINED IN A SPECIAL LINEAR COMPLEX*

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1. *Introduction.* In a series of papers by Snyder,† Carroll,‡,§ and the author, ||,¶ involutorial transformations were defined by means of a correspondence between the surfaces of a pencil and the points of a rational curve. The purpose of this paper is to apply similar methods to certain non-involutorial transformations.

2. *Definition of the Transformation.* Given a line d and two pencils of surfaces $|F_n|$ and $|F'_{n'}|$ of orders n and n' which contain d as an $(n-1)$-fold and $(n'-1)$-fold line, respectively. Make the surfaces of each pencil projective with the points of d. A point P will determine a unique surface F_n passing through it, hence a unique point O on d and a unique surface $F'_{n'}$. The line PO cuts $F'_{n'}$ in one point P' (other than O) which is defined as the image of P.

Since P and P' lie on a line which intersects d, any plane through d is transformed into itself. We shall find the plane transformation in an arbitrary plane through d and then generate the space transformation by revolving the plane about d.

3. *The Plane Transformation.* The intersections of an arbi-

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† Virgil Snyder, *On a series of involutorial cremona transformations of space defined by a pencil of ruled surfaces*, Transactions of this Society, vol. 35 (1933), pp. 341–347.

‡ Evelyn Carroll, *Systems of involutorial birational transformations contained multiply in special linear line complexes*, American Journal of Mathematics, vol. 54 (1932), pp. 707–717.

§ Evelyn Carroll-Rusk, *Cremona involutions defined by a pencil of cubic surfaces*, American Journal of Mathematics, vol. 56 (1934), pp. 96–108.

|| Amos Black, *Types of involutorial space transformations associated with certain rational curves*, Transactions of this Society, vol. 34 (1932), pp. 795–810.

¶ Amos Black, *Types of involutorial space transformations associated with certain rational curves—composite basis curves*, this Bulletin, vol. 40 (1934), pp. 417–420.

trary plane through d with $|F_n|:d^{n-1}$ and $|F'_{n'}|:d^{n'-1}$ are two pencils of lines $|l|:V$ and $|l'|:V'$, respectively, where V and V' are the vertices of the pencils. The lines of each pencil are projective with the points of d. Thus a point P determines a line l through it, a point O on d, and a line l'. The point P', the image of P, is the intersection of PO and l'.

The directions through V are perspective with the points of d which are projective with the lines $|l'|$. The conic c_2' of intersection of these two projective pencils is the image of V.

Since there is a (1, 1) correspondence between the lines of $|l|$ and the points of d, there are two coincidences, hence two lines l_3, l_4 which pass through their respective associated points O_1, O_2. Let us call the associated lines of $|l'|$, l_1' and l_2', respectively. If we choose P as an arbitrary point on l_3, its image is P_1', the point of intersection of l_3 and l_1'. If we choose P at O_1 the line PO_1 is indeterminate. However, we may readily find the image of O_1 from the inverse transformation. If we choose P' as an arbitrary point on l_1', then $P'O_1$ intersects l_3 in O_1. Thus O_1 has for image the whole line l_1'. Hence $l_3: VO_1P_1' \sim c_2' l_1' P_1'$. Similarly $l_4: VO_2P_2' \sim c_2' l_2' P_2'$. Beginning with V' and $|l'|$, in a similar manner we find $l_3': V'O_1'P_1 \sim c_2l_1P_1$ and $l_4': V'O_2'P_2 \sim c_2l_2P_2$.

There are no other fundamental points or curves, hence the plane transformation is the well known $T_3: 1^24^1$.

$$
\begin{array}{ll}
T_3: V^2O_1O_2P_1P_2, & T_3': V'^2O_1'O_2'P_1'P_2', \\
J_6 = c_2l_1l_2l_3l_4, & J_6' = c_2'l_1'l_2'l_3'l_4', \\
V \sim c_2': VV'O_1'O_2'P_1'P_2', & V' \sim c_2: VV'O_1O_2P_1P_2, \\
O_1 \sim l_1': V'P_1', & O_1' \sim l_1: VP_1, \\
O_2 \sim l_2': V'P_2', & O_2' \sim l_2: VP_2, \\
P_1 \sim l_3': V'O_1'P_1, & P_1' \sim l_3: VO_1P_1', \\
P_2 \sim l_4': V'O_2'P_2, & P_2' \sim l_4: VO_2P_2'.
\end{array}
$$

The invariant curve is the conic k_2, the intersection of the two projective pencils $|l|$, $|l'|$. Hence $k_2: VV'P_1P_2P_1'P_2'$.

4. *The Space Transformation.* Since the space transformation is generated by the plane T_3, except for orders, we can immediately determine the surfaces of the space transformation. As

we revolve the plane of T_3 about d, the points O_1, O_2 generate d. Hence l_1', l_2' generate a surface L' which is the image of d. The points P_1, P_2 generate a curve δ, whose image X' is the locus of l_3', l_4'. The vertex V generates the residual basis curve γ of the pencil $|F_n|$, and the image Γ' of γ is the locus of c_2'. In a similar manner δ', L, X, Γ are obtained. The locus of k_2 is the invariant surface K.

Since the sections of all the surfaces of the space transformation with a plane through d are known, except for the multiplicity of d, whatever the values of n and n' may be, there is no advantage in having n and n' large. We shall set up the analytic work for $n=n'=2$.

5. *Equations of the Transformation.* Let the equations of d be $x_1=0, x_2=0$, and let the coordinates of a point on d be given by $O(0, 0, \lambda, 1)$. Also let

$$|H(x)| = H_1(x) - \lambda H_2(x) = 0, \tag{1}$$

$$|H'(x)| = H_1'(x) - \lambda H_2'(x) = 0, \tag{2}$$

where

$$H_1(x) = x_1u_1(x) + x_2u_2(x), \qquad H_2(x) = x_1v_1(x) + x_2v_2(x),$$

$$H_1'(x) = x_1u_1'(x) + x_2u_2'(x), \qquad H_2'(x) = x_1v_1'(x) + x_2v_2'(x),$$

$$u_i(x) = \sum a_{ij}x_j, \qquad v_i(x) = \sum b_{ij}x_j,$$

$$u_i'(x) = \sum a_{ij}'x_j, \qquad v_i'(x) = \sum b_{ij}'x_j,$$

for $i=1, 2$ and $j=1, 2, 3, 4$. Since the parameters of the points of d and the surfaces of (1) and (2) are identical, the forms are projective.

The surface of (1) passing through $P(x)$ gives $\lambda=H_1/H_2$, where $H_1=H_1(x)$ and $H_2=H_2(x)$, and the associated point on d is $O(z)=(0, 0, H_1, H_2)$, and the associated surface of (2) is

$$H_2H_1'(x) - H_1H_2'(x) = 0. \tag{3}$$

The coordinates of any point on the line PO are given by

$$x_i' = \rho x_i + z_i, \qquad (i = 1, 2, 3, 4). \tag{4}$$

Substituting from (4) in (3) gives $\rho=L_5/K_4$, where

$$
\begin{aligned}
L_5 = {} & H_2^2(a'_{14}x_1 + a'_{24}x_2) + H_1H_2(a'_{13}x_1 + a'_{23}x_2 \\
(5) \qquad & - b'_{14}x_1 - b'_{24}x_2) - H_1^2(b'_{13}x_1 + b'_{23}x_2), \\
K_4 = {} & H_2H'_1 - H_1H'_2 .
\end{aligned}
$$

The equations of the transformation are

$$
(6) \qquad T_6: \quad x'_i = x_iL_5 - K_4z_i, \qquad (i = 1, 2, 3, 4).
$$

It is evident from the form of the equations of T_6 that $K_4=0$ is the surface of invariant points, and that $L_5=0$ is the image of d.

Any arbitrary plane π through d is the tangent plane of two surfaces of $|H'(x)|$ at points on d. The two points of contact are the points O'_1, O'_2 of the plane T_3 in π. The intersections of π with the two surfaces of $|H'(x)|$ associated with O'_1, O'_2 are the lines l'_3, l'_4 and the intersections with the associated surfaces of $|H(x)|$ are the lines l_1, l_2. Since L_5 is the locus of l_1, l_2, then the elimination of λ between the equations of the tangent plane of a surface $H'(x)$ at its associated point O and the associated surface $H(x)$ gives L_5.

The tangent plane of $H'(x)$ at its associated point O is

$$
(7) \qquad
\begin{aligned}
& [a'_{13}\lambda + a'_{14} - \lambda(b'_{13}\lambda + b'_{14})]x_1 \\
& \quad + [a'_{23}\lambda + a'_{24} - \lambda(b'_{23}\lambda + b'_{24})]x_2 = 0.
\end{aligned}
$$

The elimination of λ between (7) and (1) gives L_5. Similarly, since the locus of l'_3, l'_4 is X'_5, then the elimination of λ between (7) and (2) gives X'_5, or

$$
(8) \qquad
\begin{aligned}
X'_5 = {} & H_2'^2(a'_{14}x_1 + a'_{24}x_2) + H'_1H'_2(a'_{13}x_1 + a'_{23}x_2 \\
& - b'_{14}x_1 - b'_{24}x_2) - H_1'^2(b'_{13}x_1 + b'_{23}x_2).
\end{aligned}
$$

Since any quadric of $|H'(x)|$ is transformed into the associated quadric of $|H(x)|$, the equation of the image of γ'_3 may be found by transforming $H'_1(x)$. Hence $H'_1(x) \sim H_1L_5\Gamma_5$, where

$$
(9) \qquad
\begin{aligned}
\Gamma_5 = {} & [H'_2(a'_{14}x_1 + a'_{24}x_2) - H'_1(b'_{14}x_1 + b'_{24}x_2)]H_2 \\
& + [H'_2(a'_{13}x_1 + a'_{23}x_2) - H'_1(b'_{13}x_1 + b'_{23}x_2)]H_1.
\end{aligned}
$$

In a similar manner we may write the equations of

$$
(10) \qquad T'_6: \quad x_i = x'_iL'_5 + K_4z'_i, \qquad (i = 1, 2, 3, 4),
$$

where $(z'_i) = (0, 0, H'_1, H'_2)$ and L'_5, X_5, Γ'_5 are found from L_5, X'_5, Γ_5 by interchanging primes and unprimes.

The multiplicates of γ_3, γ_3', δ_8, δ_8' on the various surfaces are known from T_3, and the multiplicity of d can easily be verified from equations (5) to (10). Of the three tangent planes of any homoloid at any point on d, one tangent plane is the tangent plane of the associated quadric at the point. Thus the homoloidal surfaces have simple contact along d. Collecting these results, we have

$$\text{a plane} \overset{T}{\sim} S_6: d^{3+1t}\gamma_3^2\delta_8, \qquad \text{a plane} \overset{T'}{\sim} S_6': d^{3+1t}\gamma_3'^2\delta_8',$$
$$d \sim L_5: d^3\gamma_3^2\delta_8, \qquad d \sim L_5': d^3\gamma_3'^2\delta_8',$$
$$\gamma_3' \sim \Gamma_5: d^3\gamma_3\gamma_3'\delta_8, \qquad \gamma_3 \sim \Gamma_5': d^3\gamma_3\gamma_3'\delta_8',$$
$$\delta_8' \sim X_5: d^3\gamma_3^2\delta_8', \qquad \delta_8 \sim X_5': d^3\gamma_3'^2\delta_8,$$
$$K_4: d^2\gamma_3\gamma_3'\delta_8\delta_8'.$$

6. *Special Cases.* A pencil of quadrics whose base is a line and a twisted cubic contains two cones. In general, the vertex of the cone is not at its associated point in our projectivity. Whenever the vertex and the associated point coincide we have a special case.

CASE 1. Suppose $b_{13}' = b_{23}' = 0$. Then $H_2'(x)$ is a cone with vertex $O_1'(0, 0, 1, 0)$, and from (2), we see that the vertex and associated point coincide. In the pencil $|l'|$ of the plane T_3, one line is a generator of the cone, hence passes through its associated point O_1'. The image of O_1' is the line l_1 of H_2 lying in this plane. The image of the generator is P_1, the point of intersection of the generator and l_1. But as the plane is rotated about d, the point O_1' remains fixed, l_1 generates H_2, and P_1 generates a twisted cubic δ_3. Thus, in space, $O_1' \sim H_2$, and $H_2' \sim \delta_3$. However, the other incidence point O_2' varies and generates d, and P_2 varies and generates a rational curve δ_5. It is easily seen from (5) that if $b_{13}' = b_{23}' = 0$, then $L_5 = H_2L_3$, and from (8) that $X_5' = H_2'X_3'$. Then, in this special sense, in T_6' an isolated fundamental point O_1' with image H_2 is added, and in T_6 the fundamental curve δ_8 is composite with $\delta_3 \sim H_2'$ and $\delta_5 \sim X_3'$.

CASE 2. If a cone of $|H'(x)|$ has its vertex at its associated point and also a cone of $|H(x)|$ has its vertex at its associated point, but the vertices are distinct, then we have the specialization of Case 1 in both transformations. That is, both T_6 and T_6' have an isolated fundamental point on d, and both δ_8 and δ_8' are composite.

CASE 3. Let $|H(x)|$ and $|H'(x)|$ both have a cone with vertex at its associated point, and let the vertices coincide. Let the associated point be $O(0, 0, 1, 0)$ and let the cones be H_2 and H_2'. Then $b_{13}=b_{23}=b'_{13}=b'_{23}=0$. In the plane T_3 this means that O_1, O_1', P_1, P_1' all coincide with O and that $l_1=l_3$, $l_1'=l_3'$. In space $L_5=H_2L_3$, $X_5=H_2X_3$, $L_5'=H_2'L_3'$, $X_5'=H_2'X_3'$.

Since O_2, O_2', P_2, P_2', V, V' are variable, then as the plane of T_3 is revolved about d, these points generate d, d', δ_5, δ_5', γ_3, γ_3', respectively, and their respective images are L_3', L_3, X_3', X_3, Γ_5', Γ_5. The point O is fixed and $O\overset{T}{\sim}H_2'$ and $O\overset{T'}{\sim}H_2$. The two cones H_2 and H_2' intersect in d and in three other lines l_i, $(i=1, 2, 3)$. Each of these lines is parasitic; hence $\delta_3=\delta_3'=3l$.

CASE 4. Let $|H'(x)|$ have two cones with their vertices at their associated points. Let the two cones be H_2' with vertex O_1' $(0, 0, 1, 0)$ and H_1' with vertex O_2' $(0, 0, 0, 1)$. Then we have $b'_{13}=b'_3=a'_{14}=a'_{24}=0$. We find $L_5=H_1H_2\pi$, $X_5'=H_1'H_2'\pi$, where $\pi=a'_{13}x_1+a'_{23}x_2-b'_{14}x_1-b'_{24}x_2$. Then the equations of T_6 may be written in the form

$$x_i' = x_iH_1H_2\pi - K_4z_i, \qquad (i = 1, 2, 3, 4). \tag{11}$$

As we would expect from Case 1, $O_1'\sim H_2$, $O_2'\sim H_1$, $H_2'\sim\delta_3$, $H_1'\sim\bar{\delta}_3$.

As we rotate the plane T_3 about d the points O_1', O_2' remain fixed, so apparently no other points of d are fundamental, and hence d has no image; yet from the form of equations (11) the plane π is the image of d. The plane π is the tangent plane of every surface of $|H'(x)|$ at its associated point. Hence in the plane T_3 in plane π, every line of $|l'|$ passes through its associated point. Therefore, in plane π, $d\overset{T'}{\sim}\pi$ and $\pi\overset{T}{\sim}\delta_2$, where δ_2 is the conic of intersection of the two projective pencils in π.

Then in T_6' we introduce two isolated fundamental points, O_1' with image H_2 and O_2' with image H_1, and $d\sim\pi$; while in T_6 the curve δ_8 becomes composite, consisting of δ_3, $\bar{\delta}_3$, δ_2 with images H_2', H_1', π, respectively.

We may have combinations of Cases 1, 3, 4 just as Case 2 was a combination of a Case 1 with a Case 1.

7. *Special Cases due to Composite Basis Curves.* The only way in which γ_3 and γ_3' can be composite is that the composite curve consist of a line, or lines, meeting d and a residual curve.

CASE 5. Let $\gamma_3=\gamma_1\gamma_2$, where $\gamma_1: x_1=0$, $x_3=0$ and $[\gamma_1, d] =(0, 0, 0, 1)$. Then $a_{22}=b_{22}=a_{24}=b_{24}=0$, and $\Gamma_5'=y_1\Gamma_4'$, $X_5=y_1X_4$. No other surfaces of either T_6 or T_6' are composite.

Let us consider the plane T_3 in the plane $y_1=0$. The lines γ_1, d form the complete intersection of any H of $|H(x)|$ with the plane. Call the associated point of the composite H, of which y_1 is a component, $\bar{O}$. Fix a point P on γ_1. The line $P\bar{O}$ meets $l_{\bar{O}}'$ in a point P', the image of P. As $\bar{O}$ traces d, P' generates a conic $c_2: PV'\bar{O}_1'\bar{O}_2'$. As P traces γ_1, c_2 generates the plane y_1. Hence $\gamma_1\sim$ the plane. Consider an arbitrary point Q of the plane. The H determined by Q is composite and $\bar{O}$ is the associated point. The line $Q\bar{O}$ meets $l_{\bar{O}}'$ in Q', the image of Q. Then the whole line $Q\bar{O}$ transforms into Q', and the whole plane transforms into $l_{\bar{O}}'$. Thus the curve δ_8' is composite and consists of $\delta_1'=l_{\bar{O}}'$ and a δ_7'.

If γ_3 consists of a line γ_1 meeting d and a conic γ_2, then necessarily δ_8' consists of a δ_7', and a line δ_1' in the plane of γ_1, d; and the plane is a principal surface for both T_6 and T_6'.

CASE 6. If γ_3 consists of two lines each meeting d, and a third line skew to d, then the two planes determined by the two lines and d are principal for both T_6 and T_6', and δ_8' consists of a δ_6' and two lines, one in each of the principal planes.

CASE 7. If $\gamma_3=\gamma_1\gamma_2$ and $\gamma_3'=\gamma_1'\gamma_2'$, but γ_1 and γ_1' lie in different planes, then each plane is principal in both T_6 and T_6', and $\delta_8'=\delta_7'\delta_1'$ and $\delta_8=\delta_7\delta_1$.

CASE 8. If $\gamma_3=\gamma_1\gamma_2$, $\gamma_3'=\gamma_1'\gamma_2'$ and γ_1, γ_1' lie in the same plane but meet d in distinct points, then the result is the same as in Case 7 except that $\gamma_1=\delta_1$ and $\gamma_1'=\delta_1'$.

The restriction that γ_1, γ_1' meet d in distinct points is not necessary. If they meet d in the same point, or even coincide, the associated points $\bar{O}$, $\bar{O}'$ of the composite quadrics are, in general, distinct, hence the results are the same.

There may be special cases due to combinations of cases of §6 with cases of §7, but nothing new is added.

The line d may be replaced by a conic r_2 or a twisted cubic r_3 with the restriction that transformations exist for $n=n'=2$ only. However, the transformations are of the same form as the above, and no new results are obtained.

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