BROWN AND SHOOK—PLANETARY THEORY

Planetary Theory. By E. W. Brown and C. A. Shook. Cambridge, University Press, 1933. xii+229 pp.

The reviewer's lot is not "a 'appy one". He is expected to give in a brief measure of time and space a critical examination of a treatise covering several hundred pages and representing many years of investigation by renowned authors. And above all he is expected to make some contribution to the errata of the text! Realizing the un'appiness of the lot before him, the reviewer proceeds with much appreciation and with no little trepidation in an effort to summarize rather than to examine critically. And he has no contribution to make to the errata! The text already contains a page of such, contributed by experts, and that is sufficient for any volume.

When William Thomson (afterwards Lord Kelvin) and his older brother James accompanied their father on a summer tour through Germany, it is recorded that William, then 16 years of age, took Fourier's famous *Analytical Theory of Heat* for light reading. The reviewer does not recommend *Planetary Theory* for any such journey except possibly to those with Kelvin's I.Q.

"The purpose of the volume", as stated by the authors, "is the development of methods for the calculation of the general orbit of a planet". It does not aim to be a substitute for a treatise like Tisserand's or Laplace's nor does it contain detailed accounts of such classical theories as are to be found in Newcomb or elsewhere.

The mathematical processes are largely formal. Rigor is desirable when attainable, but results are much more to be desired (compare *Proverbs* on a *good name* and *riches*), particularly when the results appear to be reasonable and "useful for the prediction of physical phenomena".

Considerable portions of the volume are new, particularly the work on resonance and the Trojan asteroids.

Various forms of the equations of motion are derived in Chapter 1. For the development of planetary theory the *osculating plane* possesses certain advantages as a principal plane of reference. It is the plane passing through the sun and tangent to the orbit of the planet. Its motion, being either slow or small, affects the motion of the planet in a way which can be quite simply accounted for or neglected entirely. Two systems of coordinates may be used with the osculating plane as the principal plane of reference. Three variables are used in one system and six in the other. In the latter case the variables are the elements of the osculating ellipse. Polar coordinates are next used. The equations are then put into canonical form and also derived first with the true orbital longitude and later with the disturbed eccentric anomaly as the independent variable. The chapter concludes with the derivation of the equations of motion referred to the coordinates of the disturbing planet and also to any arbitrary plane of reference.

Devices are treated in Chapter 2 to simplify the expansions of various functions into sums of periodic terms. Lagrange's well known theorem for the expansion of a function defined by an implicit equation is considered and extended to three variables. Fourier series, functions of Fourier series, Bessel functions of the first kind, and hypergeometric series are next treated. The chapter concludes in an encouraging way with devices to relieve fatigue in calculation with slowly converging series.

The third chapter deals with elliptic motion in which the eccentricities are sufficiently small for numerical calculation without too much labor. Instead of using the three anomalies directly, namely, the true, the eccentric, and the mean, which are denoted by f, X, and g, respectively, use is made of

$$\phi = e^{ig}, \ \chi = e^{if}, \ \psi = e^{i\overline{X}}, \quad i = \sqrt{-1}.$$

Fourier developments are obtained for the radius or powers of the radius in terms of ϕ, χ, ψ, and also f and g. In certain of these expansions Bessel functions or hypergeometric series occur. The chapter concludes with certain detailed developments for g, f, r to be used for reference, and with numerical developments by harmonic analysis.

Chapter 4 deals with the development of the disturbing function, R. When the angle, I, between the orbital planes of the two planets is zero and the eccentricities e, e' are so small that they can be neglected, R can readily be expanded as a Fourier series in the coefficients of which occur powers of $\alpha = a/a'$, a, a' being the mean distances of the planets from the sun. When e, e', I are not neglected, R can be expanded in powers of these parameters, and these expansions converge rapidly as e, e', I are usually small. Difficulty occurs, however, when through integrations small denominators arise sometimes involving discontinuities.

In the older methods the time is used as the independent variable, which requires the disturbing function to be expressed in terms of the three anomalies. The methods developed in this chapter involve the theorem

$$F(p, x) = p^D F(x), \ D = x\frac{d}{dx},$$

where F is expansible in integral powers of its argument, and p^D is expansible in positive integral powers of D. By the use of this theorem, expansions for the distances between the planets are obtained in powers of the eccentricities and multiples of the true anomalies; and also in powers of the inclination and multiples of the true anomalies. Transformations are then made to obtain similar developments in multiples of the mean anomalies. To obtain the development in terms of the eccentric anomalies use is made of a hypergeometric series in which certain of the elements are linear functions of the symbolic operators D, B, B', where

$$D = \alpha\frac{\partial}{\partial\alpha}, \quad B = \chi\frac{\partial}{\partial\chi}, \quad B' = \chi'\frac{\partial}{\partial\chi'}.$$

Newcomb gave a somewhat similar expansion in terms of the eccentric anomalies of which certain portions were carried out to the seventh order with respect to the eccentricities, but he did not obtain a general formula like the one developed here which permits any coefficient to be written down at once.

The chapter ends with certain devices to abbreviate and to check the computation. For example, it is shown that after transformations have been made so as to calculate easily and readily two coefficients in a series, all the remaining coefficients and their derivatives can be deduced from these two by the use of finite formulas.

The fifth chapter contains the elements of the theory of canonical variables so far as it is needed in the later work. New symbols d, δ are suitably defined, and on making use of them the equations

$$\frac{dx_i}{dt} = \frac{\partial H_i}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial H_i}{\partial x_i}, \qquad i = 1, 2, \cdots n)$$

can be expressed in the form

$$\sum_i (dx_i\, \delta y_i - dy_i\, \delta x_i) = dt\delta H.$$

The Jacobian transformation can then readily be made from one set of variables to another set. It is shown that a general solution of the canonical equations written above is provided by the equations

$$y_i = \frac{\partial S}{\partial x_i}, \quad p_i = \frac{\partial S}{\partial q_i},$$

where p_i and q_i are arbitrary constants and S is an integral of the partial differential equation

$$H\left(x_i, \frac{\partial S}{\partial x_i}, t\right) + \frac{\partial S}{\partial t} = 0.$$

Applications are made showing various sets of canonical elements of elliptic motion.

In Chapter 6 it is shown how the theory of canonical equations can be applied to the calculation of the orbit of a planet. "All the methods depend fundamentally on the assumption that the variables differ from constants by amounts which have as factor the ratio of the disturbing mass to that of the primary, and therefore that the variables may be developed in powers of this ratio." Delaunay's method is adopted, namely, to obtain a change of variables such that the new variables are more nearly constant than the old ones. The main difference is that while Delaunay made numerous changes, the present authors show that, in general, one change is sufficient for the solution of the majority of planetary problems. "Much of the discussion in the chapter hinges on the amount of labour which the development and solution of the equations for the new variables require."

It is first shown how the work may be so adapted that use can be made of the developments for R in Chapter 4 which are not in terms of canonical elements. The disturbing function is then split up into two parts $R = R_t + R_c$, and when these are suitably defined, differential equations are obtained which have the same form as the original equations except that the terms in R_t have disappeared. Two choices for the division of R are made. In one R_c contains only those terms which produce the so-called secular motion of the elements. In the other, R_t contains the short period terms only while R_c contains the long period and secular terms. It is assumed that R_c is expansible in powers of

m', the mass of the disturbing planet, and the solutions of the above-mentioned differential equations are readily obtained to the first degree of approximation. These solutions are called the *perturbations of the elements.* The secular, periodic, and long-period terms in the first approximation are next considered.

In the calculation of the values of the variables to the second degree of approximation, special attention is given to discover what classes of terms will give sensible changes to the expressions found in the first approximations.

At the conclusion of the chapter the advantages and disadvantages of the theory which has been developed are pointed out. It has a "simplicity of analytical form" which makes it attractive for many theoretical investigations, particularly those concerned with resonance, but because of the labor involved and the slow convergence of certain series, it is doubtful if it is the most convenient for the calculation of ordinary planetary perturbations.

The direct calculation of the coordinates is developed in Chapter 7 with the true orbital longitude as the independent variable. The solutions of the equations of motion are obtained by continued approximation. The first approximation with $m'=0$ gives elliptic motion and the solution thus found is called the *elliptic approximation.* To obtain the next approximation, called the *first approximation,* the expressions found for the elliptic approximation are substituted in the terms having m' as a factor and the resulting equations are solved, giving the *second approximation.* Higher approximations can be obtained by continuing the process, but it is rarely necessary in planetary problems to proceed beyond the first approximation except for those terms which on account of their long periods have received small denominators through integration.

After the first approximations are obtained the "equations for the variation of the elements" are considered before the second approximations are treated. In the variation of the elements three new variables are introduced for two of the old variables. This procedure gives equations more convenient for calculation and furnishes an important theorem concerning the secular terms. In this way separate calculations can be made of the effects due to the secular terms and to the periodic terms in the first approximations. So as to compare the theory developed in this chapter with other theories, particularly the theory developed by Hansen, a transformation to the time as the independent variable is made. Approximate formulas for the perturbations, final definitions, and the determination of the constants conclude the chapter.

"Chapter 8 contains an attempt to place the theory of resonance on a general basis in a form which permits of application to specific problems. It appears to give a method of approach to the consideration of the question of general stability of the orbits of the planets."

Resonance is defined as a case of motion in which a particle or body, moving or capable of moving with periodic motion, is acted on by an external force whose period is the same as that of the motion of the body. The usual illustration is the equation

$$\frac{d^2x}{dt^2} + n^2x = m \sin n't,$$

which yields a Poisson term in t explicitly when $n'=n$. It is pointed out, however, that the illustration is defective inasmuch as the particular integral has a very large coefficient when n' is near n. Further, in actual mechanical problems, x does not always occur linearly and, in addition, it is sometimes present in the expression for the disturbing forces.

In order to illustrate the principal features of certain of the resonance problems, use is made of the motion of a pendulum which can make complete revolutions about a horizontal axis as well as oscillate about the vertical and which is disturbed by a periodic force. In certain cases, however, the analogy breaks down and special devices have to be employed to find out whether resonance is possible. The general case of resonance in the perturbation problem is next considered, and while the investigation given does not prove the existence of resonance it at least shows that so far no condition preventing it has appeared. The chapter concludes with a method of procedure applicable to certain of the actual cases of resonance in the solar system, particularly in the case of the asteroids disturbed by Jupiter and Saturn.

Chapter 9 is devoted to a consideration of the motion of the Trojan group of asteroids. These are two groups which are to be found at the vertices of the equilateral triangles described on the line joining the sun and Jupiter as base and in the plane of Jupiter's orbit. They oscillate about these triangular points of libration in much the same way as their mythological namesakes circulated about the walls of ancient Troy. The libration points themselves were first discovered by Lagrange in 1772 and were considered by him as "pure curiosities". The first asteroid in the group was discovered in 1901 and the last one, the tenth, in 1932.

The development in this chapter is much the same as that used in Chapter 6. The disturbing function R is expressed in powers of the eccentricities and mutual inclination. New variables are then introduced which leave the equations of motion canonical and at the same time eliminate the short period terms from R. Solutions for these variables are then obtained in terms of the time.

Finally, the perturbations due to Saturn are considered. In the ordinary planetary theory the procedure would consist in finding the perturbations due to each planet separately, then those due to their combined action, and adding the results. This method is not applicable in this case, however, inasmuch as the action of Jupiter cannot be neglected even in finding a first approximation of the direct action of Saturn. It is assumed that the mutual perturbations of Jupiter and Saturn on each other are completely known. Hence, when considering the direct effect of Jupiter upon the asteroid, the indirect effect of Saturn in causing Jupiter to deviate from elliptic motion must also be taken into account, and likewise the indirect effect of Jupiter upon Saturn. It is shown that Jupiter so alters the direct effect of Saturn upon the asteroid that by far the largest part of the action of Saturn is indirectly through its perturbation on Jupiter.

The appendix on *Harmonic Analysis* contains formulas for application to the development of a given function in a form ready for actual use.

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