ON THE NOTION OF REGULARITY OF METHODS OF SUMMATION OF INFINITE SERIES

BY J. D. TAMARKIN*

Let $\mathfrak{A}=(a_{mn})$ be the matrix of a method of summation which consists in replacing a given sequence $x=(x_1, x_2, \cdots)$ by its *transform*

$$(1) \qquad y = (y_1, y_2, \cdots), \quad y_m = \sum_{n=1}^{\infty} a_{mn}x_n, \qquad (m = 1, 2, \cdots),$$

and defining the generalized limit of x_n as $\lim_{m\to\infty} y_m$, provided this limit exists.

The method $\mathfrak{A}$ is called regular if every convergent sequence $x=\{x_n\}$ is transformed into a convergent sequence $y=\{y_m\}$ with the same limit. Necessary and sufficient conditions for regularity of $\mathfrak{A}$ are too well known to be restated here. An essential point in the whole theory is the assumption that the sequence $y\equiv y(x)$ is determined by formula (1) for *each* convergent sequence x. A (quite trivial) gain in generality may be achieved by demanding that not all the terms y_m of the sequence $\{y_m\}$ have to be considered but only those for which $m\geqq m_0$, where m_0 is a fixed integer. Even this requirement is not at all necessary, however, for the possibility of evaluating $\lim_{m\to\infty} y_m$, for which we have to know only *almost all* terms of the sequence $\{y_m\}$, that is all y_m, $m\geqq m'$, where m' need not be fixed, but on the contrary, may depend on the sequence x.

Thus we are naturally led to the following apparently less restrictive definition of regularity of the method of summation $\mathfrak{A}$.

The method of summation $\mathfrak{A}$ *is regular if* (i) *to every convergent sequence* $x=\{x_n\}$ *there corresponds an integer* $m'(x)$ *such that* y_m *as given by* (1) *exists for* $m\geqq m'(x)$, *and* (ii) *for a fixed* x,

$$\lim_{m\to\infty} y_m = \lim_{n\to\infty} x_n.$$

It turns out, however, that the modified definition of regularity

* The result of this note answers a question raised by Dr. H. Lewy.

actually is not more general than the classical one. Indeed we intend to show that if $\mathfrak{A}$ satisfies condition (i) above, then there exists a fixed number m_0 which does not depend on x, such that all the series

$$y_m = \sum_{n=1}^{\infty} a_{mn} x_n, \qquad (m \geqq m_0)$$

converge.

Assume that the number in question does not exist. Then we can find a sequence of integers $\{m_1, m_2, \cdots\}$, $m_p \to \infty$ and a set of sequences $x^{(p)} = \{x_n^{(p)}\}$, such that none of the series

$$\sum_{n=1}^{\infty} a_{mn} x_n^{(p)}, \qquad (m = m_p;\ p = 1, 2, \cdots),$$

converges. Now introduce the space (c) of all convergent sequences x. This space is a linear metric complete space if we introduce the usual definitions of the sum of two elements and of the product of an element by a scalar, and if we define the norm of x by

$$\|x\| = \sup_n |x_n|.$$

Consider the double array of linear functionals on (c) defined by

$$U_{pq}(x) = \sum_{n=1}^{q} a_{mn} x_n, \qquad (m = m_p;\ p, q = 1, 2, \cdots).$$

Our assumption is that none of the limits

$$\lim_{q \to \infty} U_{pq}(x^{(p)}), \qquad (p = 1, 2, \cdots),$$

exists. Then, by an important theorem of Banach,* the set H of elements x for which none of the limits

$$\lim_{q \to \infty} U_{pq}(x), \qquad (p = 1, 2, \cdots),$$

exists is of the second category, hence not empty. This means that there exists a convergent sequence $x = \{x_n\}$ such that no series

* S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, pp. 24–25, Theorem 4.

$$\sum_{n=1}^{\infty} a_{mn}x_n, \qquad (m = m_1, m_2, \cdots),$$

converges. This however cannot occur if $\mathfrak{A}$ is regular in the sense of the definition above, according to which all series

$$\sum_{n=1}^{\infty} a_{mn}x_n, \qquad (m \geqq m'(x)),$$

must converge.

The existence of a fixed m_0 is thus established, and at the same time it is shown that our modified definition of regularity of $\mathfrak{A}$ is equivalent to the classical one. It is clear that analogous considerations can be applied when instead of summation of series we deal with summation of integrals.

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AN EXTENSION TO POLYGAMMA FUNCTIONS OF A THEOREM OF GAUSS*

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1. *Introduction.* By the polygamma functions we mean the derivatives of log $\Gamma(x)$, that is, $\Psi(x) = \Gamma'(x)/\Gamma(x)$, $\Psi'(x), \cdots, \Psi^{(n)}(x)$.† These functions satisfy the difference equations:

$$(1)\qquad \begin{aligned} &\Psi^{(n)}(x+1) - \Psi^{(n)}(x) = (-1)^n \frac{n!}{x^{n+1}}, \\ &\Psi^{(n)}(1-x) + (-1)^{n+1}\Psi^{(n)}(x) = A_n(x), \quad A_n(x) = \frac{d^n}{dx^n}(\pi \operatorname{ctn} \pi x), \end{aligned}$$

subject to the boundary condition,

$$\Psi^{(n)}(1) = (-1)^{n+1} n! S_{n+1},$$

where we employ the abbreviation

$$S_m = 1 + 1/2^m + 1/3^m + \cdots .$$

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† The name polygamma is suggested by the paper, *Tables of the Digamma and Trigamma Functions*, by Eleanor Pairman, Tracts for Computers, No. 1, 1919.