A THEOREM ON ANALYTIC FUNCTIONS OF A REAL VARIABLE

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1. *Introduction.* Let $f(x)$ be a function of class C^∞ on $a \leqq x \leqq b$. At each point x of $[a, b]$ we form the formal Taylor series of $f(x)$,

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!} (t - x)^k.$$

This series has a definite radius of convergence, $\rho(x)$, zero, finite, or infinite, given by $1/\rho(x) = \overline{\lim}_{k\to\infty} |f^{(k)}(x)/k!|^{1/k}$. The function $f(x)$ is said to be analytic at the point x if the Taylor development of $f(x)$ about x converges to $f(t)$ over a neighborhood $|x-t| < c$, $c > 0$, of the point; $f(x)$ is analytic in an interval if it is analytic at every point of the interval.

Pringsheim stated the following theorem.*

THEOREM A. *If there exists a number $\delta > 0$ such that $\rho(x) \geqq \delta$ for $a \leqq x \leqq b$, $f(x)$ is analytic in $[a, b]$.*

However, Pringsheim's proof of the theorem is not rigorous. The purpose of this note is to establish this theorem, and, in connection with the proof, a companion theorem of considerable interest in itself.

THEOREM B. *If $\rho(x) > 0$ for $a \leqq x \leqq b$ (that is, if the Taylor development of $f(x)$ about each point converges in some neighborhood of the point), the points at which $f(x)$ is not analytic form at most a nowhere dense closed set.*

Theorem B is, in a certain sense, the best possible, since by a theorem of H. Whitney† there exist functions satisfying the

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* A. Pringsheim, *Zur Theorie der Taylor'schen Reihe und der analytischen Funktionen mit beschränkten Existenzbereich,* Mathematische Annalen, vol. 42 (1893), p. 180.

† H. Whitney, *Analytic extensions of differentiable functions defined in closed sets,* Transactions of this Society, vol. 36 (1934), pp. 63–89. I am indebted to Dr. Whitney for calling my attention to this paper.

conditions of Theorem B and having the points of an arbitrary nowhere dense closed set as singular points.

Theorem B can also be stated in the following equivalent form.

THEOREM B′. *If $f(x)$ is of class C^∞ on $[a, b]$ and analytic at no point of $[a, b]$, there must exist an everywhere dense set of points, G, on $[a, b]$ such that the Taylor development of $f(x)$ about each point of G is divergent.*

We shall need the following lemma.*

LEMMA. *If H is a perfect point set on the interval $[\alpha, \beta]$, and if $H=\sum_{n=0}^{\infty} H_n$, where the H_n are enumerable in number, there exist a value n_0 of n and a sub-interval $[\alpha_0, \beta_0]$ such that H_{n_0} is dense in $H\cdot[\alpha_0, \beta_0]$.*

2. *Proof of Theorem* B. For each x in $[a, b]$,

$$\frac{1}{\rho(x)} = \overline{\lim_{n\to\infty}} \left| \frac{f^{(n)}(x)}{n!} \right|^{1/n} < \infty .$$

This implies that there exists a finite function $\mu(x)$ such that

$$\left| \frac{1}{n!} f^{(n)}(x) \right|^{1/n} \leqq \mu(x), \qquad (n = 1, 2, \cdots),$$

or,

$$\left| f^{(n)}(x) \right| \leqq n![\mu(x)]^n, \qquad (n = 1, 2, \cdots).$$

Let E_k be the set (not necessarily non-empty) of points x such that

$$k \leqq \mu(x) < k+1, \quad (k = 0, 1, 2, \cdots).$$

It is clear that $[a, b]=\sum_{=0}^{\infty} E_k$. By the lemma, there is a sub-interval $[\alpha, \beta]$ and an integer k_0 such that E_{k_0} is dense in $[\alpha, \beta]$. For every point of $E_{k_0}\cdot[\alpha, \beta]$,

$$(1) \qquad \left| f^{(n)}(x) \right| \leqq n![\mu(x)]^n < n!(k_0+1)^n, \qquad (n = 1, 2, \cdots).$$

For every point of $[\alpha, \beta]\cdot C(E_{k_0})$, (1) holds by continuity. That

* See Lebesgue, *Leçons sur l'Intégration*, 2d ed., 1928, p. 203. See also S. Banach, *Théorie des Opérations Linéaires*, 1932, p. 14 (Theorem 2).

is, (1) holds uniformly in $[\alpha, \beta]$.* But this is a well known sufficient condition for $f(x)$ to be analytic in $[\alpha, \beta]$. The same reasoning applies to any sub-interval of $[a, b]-[\alpha, \beta]$; thus in any sub-interval there is a further sub-interval in which $f(x)$ is analytic. The points at which $f(x)$ is not analytic thus form a nowhere dense set, which is obviously closed.

3. *Proof of Theorem* A. Assume the theorem false; we shall obtain a contradiction. We have, then, a non-empty set H of points where $f(x)$ is not analytic, and by Theorem B, H is closed and nowhere dense.

We first show that H is perfect. Suppose that H contained an isolated point X. The function $f(x)$ is continuous with all derivatives at X; $f(x)$ is analytic in each of the intervals $X-h<x<X$ and $X<x<X+h$, for some $h>0$, and can be extended analytically across the point X in both directions. It follows immediately that $f(x)$ is analytic at X, so that X is not a singular point. H being perfect, from now on we shall confine our attention to an interval $[a_1, b_1]$ such that $b_1-a_1<\delta/4$ and $[a_1, b_1]$ contains a perfect subset E of H.

Since by hypothesis

$$\overline{\lim_{n\to\infty}}\left|\frac{f^{(n)}(x)}{n!}\right|^{1/n}=\frac{1}{\rho(x)}\leqq\frac{1}{\delta}$$

for every point of $[a_1, b_1]$, it follows that for each x in E there is an integer N_x such that

$$\left|\frac{1}{n!}f^{(n)}(x)\right|^{1/n}<\frac{2}{\delta}, \qquad (n\geqq N_x);$$

hence

$$|f^{(n)}(x)|\leqq n!\lambda^n, \qquad (\lambda=2/\delta,\ n\geqq N_x). \tag{2}$$

Let E_k be the set of points of E for which $N_x=k$. By the lemma, there exist a sub-interval $[\alpha, \beta]$ and a value k_0 of k such that E_{k_0} is dense in $E\cdot[\alpha, \beta]$. For x in $E_{k_0}\cdot[\alpha, \beta]$, (2) holds for $n\geqq k_0$. For x in $(E-E_{k_0})\cdot[\alpha, \beta]$, (2) holds for $n\geqq k_0$, by continuity. Thus (2) holds uniformly for x in $E\cdot[\alpha, \beta]$, $n\geqq k_0$.

* This fact can be obtained as a special case of a general theorem in the theory of operations, which is established by similar reasoning; see S. Banach, op. cit., p. 19 (Theorem 11).

Let (x_0, y_0) be a complementary interval of the nowhere dense set $E\cdot[\alpha, \beta]$. Then the Taylor series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

converges to $f(x)$ for $x_0 \leqq x < y_0$. This follows at once from the facts that $f^{(k)}(x)$ is continuous on $x_0 \leqq x < y_0$ for $k = 0, 1, 2, \cdots$, and that $y_0 - x_0 < \delta/4 < \delta$.

Define an auxiliary function $\phi(x) = (\beta_1 - \alpha)/(\beta_1 - x)$, where $\delta/4 > \beta_1 - \beta > \beta - \alpha > 0$. The function $\phi(x)$ is analytic on $[\alpha, \beta]$ and is represented over the whole of $[\alpha, \beta]$ by its Taylor development about any point of $[\alpha, \beta]$. We have

$$\phi^{(k)}(x) = \frac{(\beta_1 - \alpha)\cdot k!}{(\beta_1 - x)^{k+1}} \geqq \frac{(\beta_1 - \alpha)\cdot k!}{(\beta_1 - \alpha)^{k+1}} \geqq k!\lambda^k,$$

$$(k = 0, 1, 2, \cdots; \alpha \leqq x \leqq \beta).$$

Now form $\psi(x) = \phi(x) - f(x)$. The function $\psi(x)$ is represented by its Taylor development about x_0 for $x_0 \leqq x < y_0$; for $n \geqq k_0$, $\psi^{(n)}(x_0) \geqq 0$ and $\psi^{(n)}(y_0) \geqq 0$ by (2). Hence for $n \geqq k_0$, $\psi^{(n)}(x) \geqq 0$ for $x_0 \leqq x \leqq y_0$, since we may differentiate a power series termwise any number of times in the interior of its interval of convergence, so that $\psi^{(n)}(x)$ is represented over $x_0 \leqq x < y_0$ by a series of non-negative terms, for $n \geqq k_0$. This reasoning applies to any complementary interval of $E\cdot[\alpha, \beta]$, with the same function $\psi(x)$. Hence $\psi^{(n)}(x) \geqq 0$ for $\alpha \leqq x \leqq \beta$, $n \geqq k_0$. By a well known theorem of S. Bernstein, $\psi^{(k_0)}(x)$ is analytic for $\alpha \leqq x \leqq \beta$, and consequently $\psi(x)$ is analytic in the same interval. But then $f(x) = \phi(x) - \psi(x)$ is analytic in $[\alpha, \beta]$, contrary to the hypothesis that H was not an empty set. Hence H is an empty set, and the theorem is proved.

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