A NOTE ON TAYLOR'S THEOREM

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Let the function $f(x)$ be such that $f^{(n)}(a) \equiv d^n f(x)/dx^n$ at $x=a$ exists; then, for $|h|$ sufficiently small, we can write

$$(1)\ f(a+h)=f(a)+h\,f'(a)+\frac{h^2}{2!}f''(a)+\cdots+\frac{h^n}{n!}f^{(n)}(a)+w(a,h).$$

It is well known that $w(a, h)=o(h^n)$ as $h\to 0$,* and the more precise result that $|w(a,h)| \leqq |h^n|\,v(a,h)$, where $v(a, h)$ is the least upper bound for $0<|t|<|h|$ of

$$\left| \frac{f^{(n-1)}(a+t)-f^{(n-1)}(a)}{t} - f^{(n)}(a) \right|$$

is given by S. Pollard.†

In this note we are concerned primarily with the behavior, as $h\to 0$, of derivatives with respect to h of the function $w(a, h)$. The point a being fixed, we designate the ith such derivative, $i \geqq 0$, by $d^i w(a,h)/dh^i$. Our theorem, a generalization of Pollard's theorem, is given below.

THEOREM. *If $f(x)$ is such that $f^{(n)}(a)$ exists, then for $i=0, 1, 2, \cdots, n-1$, and $|h|$ sufficiently small*

$$\left| \frac{d^i}{dh^i} w(a,h) \right| \leqq \frac{|h^{n-i}|}{(n-i)!} v(a,h).$$

PROOF. Since

$$\frac{d^i}{dt^i} f(a+t) \equiv \frac{d^i}{dx^i} f(x)\Big]_{x=a+t} \equiv f^{(i)}(a+t),$$

* See E. W. Hobson, *The Theory of Functions of a Real Variable*, vol. 1, 3d ed., pp. 368–370. We use here the more restrictive of the two definitions given by Hobson for $f^{(n)}(x)$. The existence of $f^{(n)}(a)$ then insures the existence and continuity in an open interval containing a of all derivatives of lower order.

† S. Pollard, *On the descriptive form of Taylor's theorem*, Cambridge Philosophical Society Proceedings, vol. 23 (1926–27), pp. 383–385. Pollard's proof seems only to establish the less sharp result $|w(a,h)| \leqq n|h^n|v(a,h)$.

we see, upon writing t for h in (1) and differentiating, that (i) for $i<n$,

$$\frac{d^i}{dt^i}\, w(a, t) = o(1), \quad \text{as } t \to 0,$$

which insures that for $|t|$ sufficiently small and $j=1, 2, \cdots, n-1$,

$$\int_0^t \frac{d^j}{dt^j}\, w(a, t)dt = \frac{d^{j-1}}{dt^{j-1}}\, w(a, t);$$

and that (ii) for $|h|$ sufficiently small and $|t| < |h|$,

$$\left| \frac{d^{n-1}}{dt^{n-1}}\, w(a, t) \right| = \left| t\left[\frac{f^{(n-1)}(a+t) - f^{(n-1)}(a)}{t} - f^{(n)}(a)\right]\right|$$
$$\leqq |t|\, v(a, h).$$

We have then for $|h|$ sufficiently small

$$\left| \frac{d^i}{dh^i}\, w(a, h) \right|$$
$$= \left| \int_0^h dt_{n-i-2} \int_0^{t_{n-i-2}} dt_{n-i-3} \cdots \int_0^{t_1} \frac{d^{n-1}}{dt^{n-1}}\, w(a, t)dt \right|$$
$$\leqq \left| \int_0^h dt_{n-i-2} \int_0^{t_{n-i-2}} dt_{n-i-3} \cdots \int_0^{t_1} \left| \frac{d^{n-1}}{dt^{n-1}}\, w(a, t) \right| dt \right|$$
$$\leqq \frac{|h^{n-i}|}{(n-i)!}\, v(a, h).$$

Since $v(a, h) = o(1)$ as $h \to 0$, it follows from our theorem that for $i=0, 1, 2, \cdots, n-1$,

$$\frac{d^i}{dh^i}\, w(a, h) = o(h^{n-i}), \quad \text{as } h \to 0.*$$

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* For $i>0$ the result given here can be obtained from that for $i=0$ by comparing the expansion analogous to (1) of $f^{(i)}(a+h)$ with the equation obtained by differentiating (1) i times.