# ON THE $r$TH DERIVED CONJUGATE FUNCTION*

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1. *Introduction.* We assume throughout this note that the function $f(x)$ is Lebesgue integrable on $(-\pi, \pi)$ and of period $2\pi$; then the series

$$(1)\qquad \sum_{n=1}^{\infty}(-b_n\cos nx + a_n \sin nx),$$

where $a_n$ and $b_n$ are the Fourier coefficients, is known as the conjugate Fourier series.

It is customary to associate with the series (1) as sum or "conventionalized" sum the conjugate function of $f(x)$ which is either the limit

$$(2)\qquad \tilde{f}_1(x) \equiv \frac{-1}{2\pi}\lim_{\epsilon\to 0}\int_\epsilon^\pi \{f(x+s)-f(x-s)\}\operatorname{ctn}\frac{s}{2}\,ds,$$

or the equivalent limit

$$(3)\qquad \tilde{f}_2(x) \equiv \frac{-1}{\pi}\lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{f(x+s)-f(x-s)}{s}\,ds;$$

and to associate with the first derived series of the series (1) the first derived conjugate function which is either the limit

$$(4)\qquad \tilde{f}_1'(x) \equiv \frac{-1}{4\pi}\lim_{\epsilon\to 0}\int_\epsilon^\pi \{f(x+s)+f(x-s)-2f(x)\}\csc^2\frac{s}{2}\,ds,$$

or the equivalent limit

$$(5)\qquad \tilde{f}_2'(x) \equiv \frac{-1}{\pi}\lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{f(x+s)+f(x-s)-2f(x)}{s^2}\,ds.\dagger$$

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† For a proof of the equivalence of (2) and (3) see G. H. Hardy and J. E. Littlewood, *The allied series of a Fourier series*, Proceedings of the London Mathematical Society, vol. 24 (1925), pp. 211–246 (p. 221). A proof of the simultaneous existence of (4) and (5) is given by A. H. Smith, *On the summability of derived series of the Fourier-Lebesgue type*, Quarterly Journal of Mathematics, (Oxford Series), vol. 4 (1933), pp. 93–106 (p. 106); that they are equivalent follows from Theorem 1 of this note.

The limits (2) and (3) and, provided $f(x)$ is of bounded variation, the limits (4) and (5) exist almost everywhere.*

In a recent paper,† generalizing (2) and (4), we defined the $r$th derived conjugate function as follows.

DEFINITION 1.

$$\tilde{f}_1^{(r)}(x) \equiv \frac{(-1)^{r+1}}{2\pi} \lim_{\epsilon\to 0} \int_\epsilon^\pi A_r(s) \frac{d^r \operatorname{ctn}(s/2)}{ds^r} ds - C_r,$$

where $f^{(i)}(x)$ designates the $i$th generalized derivative of $f(x)$,

$$(6) \qquad A_r(s) \equiv f(x+s) + (-1)^{r+1} f(x-s) - 2 \sum_{i=0}^{[(r-1)/2]} \frac{s^{r-1-2i}}{(r-1-2i)!} f^{(r-1-2i)}(x),$$

and

$$(7) \qquad C_r \equiv \sum_{i=0}^{[r/2]-1} f^{(r-1-2i)}(x) \sum_{j=i}^{[r/2]-1} \frac{\pi^{2j-2i}}{(2j+1-2i)!} \left. \frac{d^{2j+1} \operatorname{ctn}(s/2)}{ds^{2j+1}} \right]_{s=\pi}.$$

We used the limit $\tilde{f}_1^{(r)}(x)$, which (i) exists wherever $f^{(r+1)}(x)$ exists and (ii) exists almost everywhere when $d^{r-1}f(x)/dx^{r-1}$ is of bounded variation, as the $N_{z_{r+1}}$ sum of the $r$th derived series of the series (1).‡

For use as a sum function in theorems concerning the summability of the $r$th derived series of the series (1) by certain standard methods, among which is the first form of the Bosanquet-Linfoot $(\alpha, \beta)$ method,§ it is desirable to define an $r$th derived conjugate function of the type (3), (5). In this note

* A. Plessner, *Zur Theorie der konjugierten trigonometrischen Reihen*, Mitteilungen des Mathematischen Seminars der Universität Giessen, Heft 10 (1923).

† A. F. Moursund, *On summation of derived series of the conjugate Fourier series*, Annals of Mathematics, (2), vol. 36 (1935), p. 184. Throughout this note $[r]$ is the greatest integer $\leqq r$. Hence $C_0$, $C_1=0$. For the definition and properties of generalized derivatives see Ch. de la Vallée-Poussin, *Approximation des fonctions par des polynomes*, Académie Royale de Belgique, Bulletins, Classe des Sciences, (1908), pp. 195–254 (p. 214).

‡ See loc. cit. The $N_{z_p}$ method includes as a special case the second form of the summation method defined by L. S. Bosanquet and E. H. Linfoot in *Generalized means and the summability of Fourier series*, Quarterly Journal of Mathematics, (Oxford Series), vol. 2 (1931), pp. 207–229.

§ Loc. cit., p. 208.

we define such an $r$th derived conjugate function, prove that the function is equivalent to $\tilde{f}_1^{(r)}(x)$, and use it as the Bosanquet-Linfoot sum of the $r$th derived series of the series (1).*

2. *Definition of* $\tilde{f}_2^{(r)}(x)$. *Theorems*. The function $A_r(s)$ is defined in the introduction only for $0 \leqq s \leqq \pi$. We now set for $s$ on $(-\infty, \infty)$

$$A_r(s) \equiv f(x+s) + (-1)^{r+1} f(x-s) - 2 \sum_{i=0}^{[(r-1)/2]} \frac{\{p(s)\}^{r-1-2i}}{(r-1-2i)!} f^{(r-1-2i)}(x), \tag{8}$$

where $p(s) = s - 2k\pi$ for $(2k-1)\pi < s \leqq (2k+1)\pi$; and we define $\tilde{f}_2^{(r)}(x)$ as follows.

DEFINITION 2.

$$\tilde{f}_2^{(r)}(x) \equiv \frac{-r!}{\pi} \lim_{\epsilon \to 0} \int_\epsilon^\infty \frac{A_r(s)}{s^{r+1}} ds - C_r.$$

The following theorems give our principal results.

THEOREM 1. *Whenever either of the limits* $\tilde{f}_1^{(r)}(x), \tilde{f}_2^{(r)}(x)$ *exists, the other exists and the two are equal.*

THEOREM 2. *The rth derived series of the series* (1) *is summable to* $\tilde{f}_2^{(r)}(x)$ *by the Bosanquet-Linfoot* $(\alpha, \beta)$ *method with* $\alpha = r$, $\beta > 1$ *or* $\alpha > r$ *wherever* $\tilde{f}_2^{(r)}(x)$ *exists and* $\int_0^s |A_r(t)| dt = o(s^{r+1})$, *as* $s \to 0$.†

3. *Proof of Theorem* 1. We use the following lemma in the proofs of both Theorems 1 and 2.

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* An $r$th derived conjugate function of the type (3), (5) is defined by A. H. Smith in a recent paper, *On the summability of derived conjugate series of the Fourier-Lebesgue type*, this Bulletin, vol. 40 (1934), pp. 406–412; however, it is impossible to give good conditions for the existence of his function. A simple example will serve to show this. Let $f(x) = \sin x$; then the second derived conjugate series is $\cos x$ and converges to 1 when $x = 0$, while for $x = 0$, Smith's $g^{(2)}(x) = \lim_{\epsilon \to 0} -(4/\pi)\int_\epsilon^\infty (\sin t/t^3) dt$ and does not exist. All further references to Smith are to this paper.

† We give this theorem primarily to illustrate the use of the function $\tilde{f}_2^{(r)}(x)$. The condition $\int_0^s |A_r(t)| dt = o(s^{r+1})$ is equivalent to the condition $\int_0^s (|A_r(t)|/t^r) dt = o(s)$; see Smith, loc. cit., pp. 411–412.

LEMMA 1. *For $s$ on $(-\pi, \pi)$ the series*

$$\sum_{-\infty}^{\infty}{}' \frac{(-1)^r r!}{(s+2k\pi)^{r+1}} \equiv \lim_{k\to\infty}\left(\sum_{-k}^{-1} + \sum_{1}^{k}\right)$$

*converges uniformly to the function*

$$\frac{1}{2}\frac{d^r \operatorname{ctn}(s/2)}{ds^r} + \frac{(-1)^{r+1}r!}{s^{r+1}}.$$

PROOF. The lemma follows upon differentiating termwise $r$ times the series $\sum_{-\infty}^{\prime\infty} 1/(s+2k\pi)$ which converges uniformly on $(-\pi, \pi)$ to the function $(1/2)\operatorname{ctn}(s/2) - 1/s$.*

PROOF OF THEOREM 1.† Since $A_r(s)$ is of period $2\pi$ and the functions

$$\frac{A_r(s)}{s^{r+1}}, \qquad A_r(s)\left\{\frac{1}{2}\frac{d^r \operatorname{ctn}(s/2)}{ds^r} + \frac{(-1)^{r+1}r!}{s^{r+1}}\right\},$$

are even functions of $s$, we have, using Lemma 1,

$$\begin{aligned}
\int_\epsilon^\infty \frac{A_r(s)}{s^{r+1}}\,ds &= \int_\epsilon^\pi + \int_\pi^\infty \\
&= \int_\epsilon^\pi \frac{A_r(s)}{s^{r+1}}\,ds + \frac{1}{2}\sum_{-\infty}^{\infty}{}' \int_{(2k-1)\pi}^{(2k+1)\pi} \frac{A_r(s)}{s^{r+1}}\,ds \\
&= \int_\epsilon^\pi \frac{A_r(s)}{s^{r+1}}\,ds + \frac{1}{2}\sum_{-\infty}^{\infty}{}' \int_{-\pi}^{\pi} \frac{A_r(s)}{(s+2k\pi)^{r+1}}\,ds \\
&= \int_\epsilon^\pi \frac{A_r(s)}{s^{r+1}}\,ds \\
&\qquad + \frac{(-1)^r}{2r!}\int_{-\pi}^{\pi} A_r(s)\left\{\frac{1}{2}\frac{d^r \operatorname{ctn}(s/2)}{ds^r} + \frac{(-1)^{r+1}r!}{s^{r+1}}\right\}ds \\
&= \int_\epsilon^\pi \frac{A_r(s)}{s^{r+1}}\,ds \\
&\qquad + \frac{(-1)^r}{r!}\int_0^\pi A_r(s)\left\{\frac{1}{2}\frac{d^r \operatorname{ctn}(s/2)}{ds^r} + \frac{(-1)^{r+1}r!}{s^{r+1}}\right\}ds
\end{aligned}$$

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* T. J. I'a. Bromwich, *Theory of Infinite Series*, 1926 edition, pp. 216–218.

† This proof was suggested to us by the proof of the equivalence of $\bar{f}_1(x)$ and $\bar{f}_2(x)$ given by Hardy and Littlewood, loc. cit., p. 221.

$$= \frac{(-1)^r}{2r!}\int_\epsilon^\pi A_r(s)\,\frac{d^r \operatorname{ctn}(s/2)}{ds^r}\,ds$$

$$+ \frac{(-1)^r}{r!}\int_0^\epsilon A_r(s)\left\{\frac{1}{2}\,\frac{d^r \operatorname{ctn}(s/2)}{ds^r} + \frac{(-1)^{r+1}r!}{s^{r+1}}\right\}ds$$

$$= \frac{(-1)^r}{2r!}\int_\epsilon^\pi A_r(s)\,\frac{d^r \operatorname{ctn}(s/2)}{ds^r}\,ds + o(1) \text{ as } \epsilon \to 0.$$

The theorem follows when we multiply by the constant factor $-r!/\pi$.

4. *Proof of Theorem* 2. We base the proof of Theorem 2 upon the proof of an analogous theorem given by A. H. Smith, and use as far as possible his notation.* The use of $\tilde{f}_2^{(r)}(x)$ instead of Smith's $g^{(r)}(x)$ as sum for the $r$th derived conjugate Fourier series complicates the proof of our theorem.†

In addition to Smith's lemmas we need the following fundamental lemma.

LEMMA 2. *For* $\beta > 0$,

$$2(-1)^r \sum_{i=0}^{[(r-1)/2]} \frac{f^{(r-1-2i)}(x)}{(r-1-2i)!}\int_0^\infty \{p(t)\}^{r-1-2i}\,\bar{\lambda}^{(r)}_{r+1,\beta}(n, t)dt$$
$$= C_r + o(1) \textit{ as } n \to \infty.$$

PROOF. Using Smith's Lemmas 1 and 2 and our Lemma 1, we have‡

$$\int_0^\infty \{p(t)\}^{r-1-2i}\,\bar{\lambda}^{(r)}_{r+1,\beta}(n, t)dt = \frac{1}{2}\int_{-\infty}^\infty \{p(t)\}^{r-1-2i}\,\bar{\lambda}^{(r)}_{r+1,\beta}(n, t)dt$$

$$= \frac{1}{2}\sum_{-\infty}^{\infty}\int_{(2k-1)\pi}^{(2k+1)\pi} (t - 2k\pi)^{r-1-2i}\,\bar{\lambda}^{(r)}_{r+1,\beta}(n, t)dt$$

$$= \frac{1}{2}\sum_{-\infty}^{\infty}\int_{-\pi}^{\pi} t^{r-1-2i}\,\bar{\lambda}^{(r)}_{r+1,\beta}(n, t + 2k\pi)dt$$

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* Loc. cit. The analogous theorem referred to is Smith's Theorem 1.

† See, however, the footnote in which Smith's paper is first cited.

‡ We assume throughout this proof and the remainder of this section that the reader is thoroughly acquainted with Smith's paper, loc. cit. As an immediate consequence of Smith's Lemma 1, his Lemma 2 holds for all $t>0$.

$$= \frac{1}{2} \sum_{k=-\infty}^{\infty} \left\{ \sum_{j=1}^{r-1-2i} (-1)^{j-1} \right.$$

$$\cdot \frac{(r-1-2i)!}{(r-2i-j)!} t^{r-2i-j} \bar{\lambda}_{r+1,\beta}^{(r-j)}(n, t+2k\pi) \Bigg]_{-\pi}^{\pi}$$

$$\left. + (-1)^{r-1-2i}(r-1-2i)! \int_{-\pi}^{\pi} \bar{\lambda}_{r+1,\beta}^{(2i+1)}(n, t+2k\pi)dt \right\}$$

$$= \frac{1}{2} \sum_{j=1}^{r-1-2i} (-1)^{j-1} \frac{(r-1-2i)!}{(r-2i-j)!} \pi^{r-2i-j}$$

$$\sum_{k=-\infty}^{\infty} \left\{ \bar{\lambda}_{r+1,\beta}^{(r-j)}\{n, (2k+1)\pi\} - (-1)^{r-j} \bar{\lambda}_{r+1,\beta}^{(r-j)}\{n, (2k-1)\pi\} \right\}$$

$$= \sum_{j=i}^{[r/2]-1} (-1)^r \frac{(r-1-2i)!}{(r-2i-j)!} \pi^{2j-2i+1} \sum_{k=-\infty}^{\infty} \left\{ \frac{-(2j+1)!}{\pi\{(2k+1)\pi\}^{2j+2}} \right.$$

$$\left. + \frac{o(1) \text{ as } n \to \infty}{\{(2k+1)\pi\}^{2j+3}} \right\}$$

$$= \frac{(-1)^r}{2} \sum_{j=i}^{[r/2]-1} \frac{(r-1-2i)!}{(r-2i-j)!} \pi^{2j-2i} \frac{d^{2j+1} \operatorname{ctn}(s/2)}{ds^{2j+1}} \Bigg]_{s=\pi}$$

$$+ o(1) \text{ as } n \to \infty .$$

When $r$ is even $[(r-1)/2] = [r/2]-1$, and when $r$ is odd $[(r-1)/2] = [r/2]$, but the term where $i = [(r-1)/2]$ vanishes. The lemma follows when we multiply by the factor $2(-1)^r f^{(r-1-2i)}(x)/(r-1-2i)!$ and sum with respect to $i$.

PROOF OF THEOREM 2. To prove our Theorem 2 we replace $\omega_r(t)$ by $A_r(t)$ in the preliminary material and proof of Smith's Theorem 1; then using Lemma 2 we see that the $n$th partial Bosanquet-Linfoot $(r, \beta)$ sum of the $r$th derived conjugate Fourier series is

$$(9) \quad J \equiv (-1)^{r+1} \int_0^{\infty} A_r(t) \bar{\lambda}_{r+1,\beta}^{(r)}(n, t)dt - C_r + o(1) \text{ as } n \to \infty .$$

Equation (9) takes the place of Smith's equation (13). To the right-hand sides of his equations (14) and (15) must be added, respectively, $C_{2r} + o(1)$ and $-C_{2r+1} + o(1)$. When modified as indicated above, Smith's proof establishes our theorem.

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